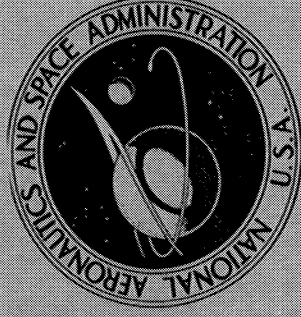


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**ADVANCED STRUCTURAL GEOMETRY STUDIES
Part II - A Geometric Transformation Concept
for Expanding Rigid Structures**

by Joseph D. Clinton

Prepared by
SOUTHERN ILLINOIS UNIVERSITY
Carbondale, Ill.
for

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FOREWORD

This final report was prepared by the School of Technology at Southern Illinois University, Carbondale, Illinois under NASA Contract NGR 14-008-002. The contract was administered by the NASA Office of Advanced Research and Technology.

Personnel participating in the research included: Julian H. Lauchner, principal investigator; Joseph D. Clinton, prime investigator; R. Buckminster Fuller, research consultant; Wayne Booth, Ann C. Garrison, Michael Keeling, Allen Kilty, Mark B. Mabee, and Richard M. Moeller, computer programmers.

PART II

A GEOMETRIC TRANSFORMATION CONCEPT FOR EXPANDING RIGID STRUCTURES

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PART III

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Computer Software Management
and Information Center

The documentation and program developed for the advanced structural geometry studies will be made available to the public through COSMIC.

COSMIC (Computer Software Management and Information Center) was established early in 1966 at the University of Georgia to collect and disseminate to the public computer software developed by government agencies. Since that time thousands of computer programs in all areas of aerospace engineering, mathematics, business, and industry have been distributed to requesters throughout the United States.

The Technology Utilization Division of NASA, designed to enlarge the return on the public investment in aeronautical and space activities, was the first government agency to participate formally. In July 1968 the Atomic Energy Commission and in November 1968 the Department of Defense joined in the COSMIC endeavor. With the addition of these two major agencies, the original concept of making tax-paid developments available to the public was expanded to make COSMIC a transfer point between and within government agencies as well.

Requests for documentation or information concerning this program should be directed to:

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The University of Georgia
Barrow Hall
Athens, Georgia 30601

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2.1 INTRODUCTION

An important area of research in structural design concepts for future aerospace applications is that of expandable structures. There are distinct and obvious advantages in the capability of developing a structural configuration that may be packaged in a small container for launch, and expanded to a predetermined size and shape. This section of the report shall propose a new and unique concept of a structural system capable of being packaged in a small compact area, and when desired, be deployed into a final, larger structural system.

A very broad meaning is implied in the term "expandable structure". It includes any structure that geometrically transforms: unidirectionally or omnidirectionally. In the area of aerospace applications generally three classifications of expandable structures are given:*

Inflated, pressure-stabilized structures:

Inflated, rigidized structures:

Mechanically expanded, framework-stabilized structures.

Attention herein will be given to the mechanically expanded, framework-stabilized structures.

*Aerospace Expandable Structures 1.

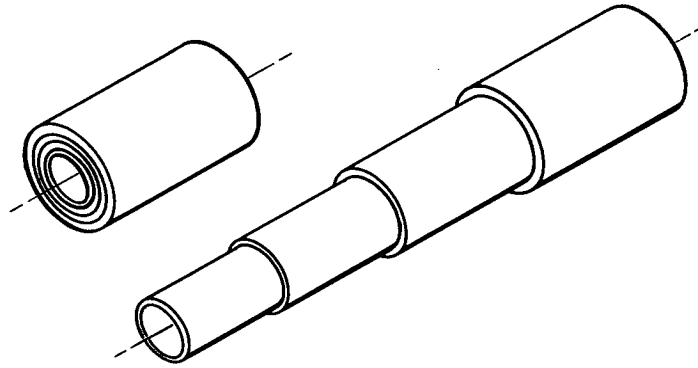
The mechanically expanded or rigid system "is made up of rigid components compacted into a small package which, upon signal, rearrange themselves to provide for greater surface area and enclosed volume."* Five basic concepts have been considered as applicable to rigid expandable structures: (one is relatively new and little in regards to application has been done.**)

- | | |
|------------------------------|------------|
| 1. Telescoping Concept | Figure 2.1 |
| 2. Folding Concept | Figure 2.2 |
| 3. Fan Concept | Figure 2.3 |
| 4. Umbrella Concept | Figure 2.4 |
| 5. Variable Geometry Concept | Figure 2.5 |

A sixth concept, "The Geometric Transformation Concept" is proposed as an expandable system and is reported herein. The geometrical transformation concept is divided into two types; the tessellation transformation concept a two-space expansion; and the polyhedral transformation concept a three-space expansion. Figure 2.6.

*Wright, F. N. 1

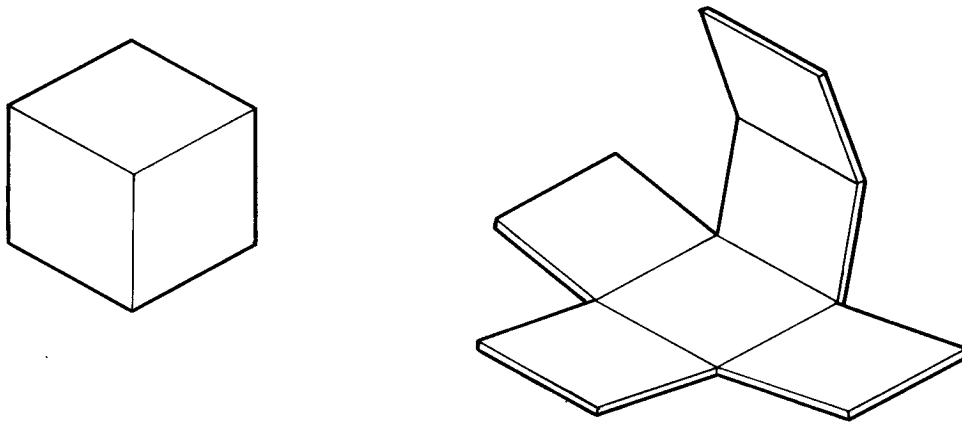
**Lebovits, M. et al; 1



1 Telescoping Concept:

Figure 2.1

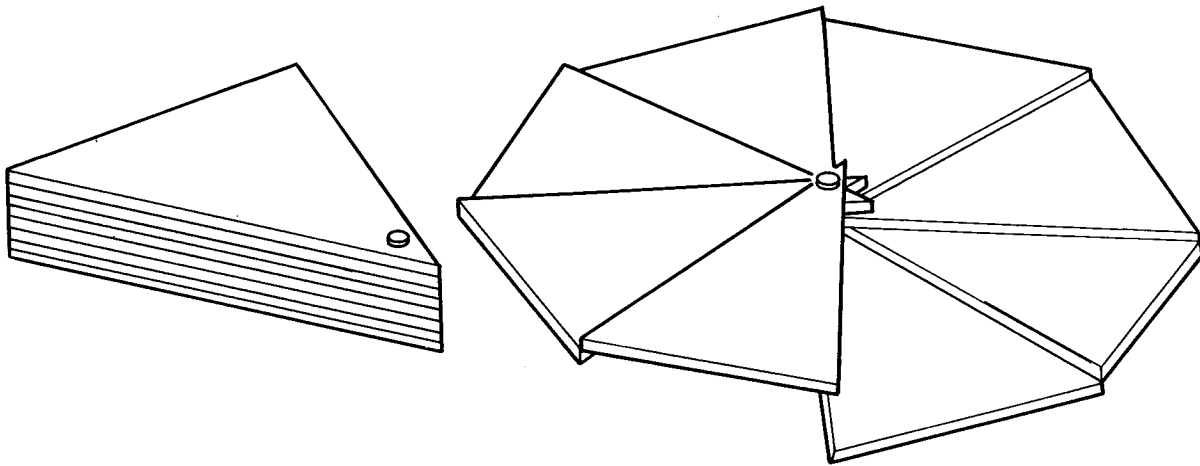
The telescoping mechanism consist of a series of rigid components that translate on a common axis sliding in and out of each other, thus permitting unidirectional expansion in packing.



2 Folding Concept:

Figure 2.2

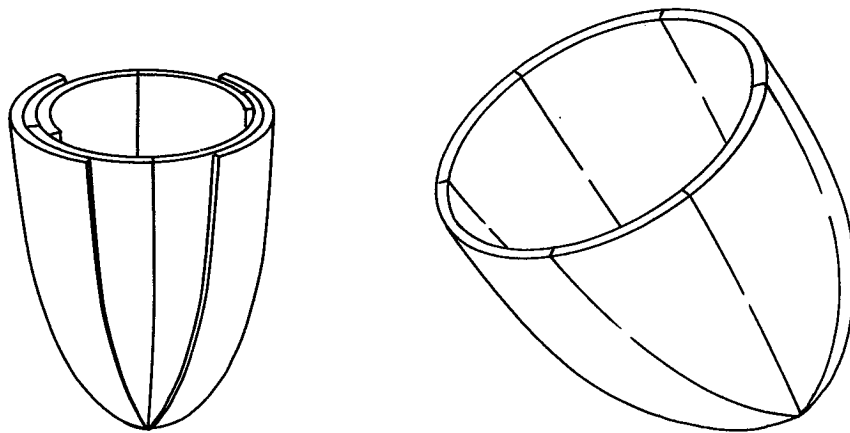
The folding mechanisms consist of a series of rigid bars and/or panels which are hinged together at the ends or sides. They are expanded by a rotation about an axis.



3 Fan Concept:

Figure 2.3

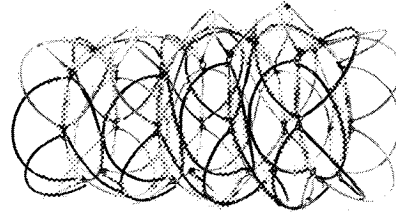
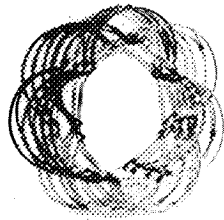
The fan concept entails rotation of rigid components in common planes about a central point.



4 Umbrella Concept:

Figure 2.4

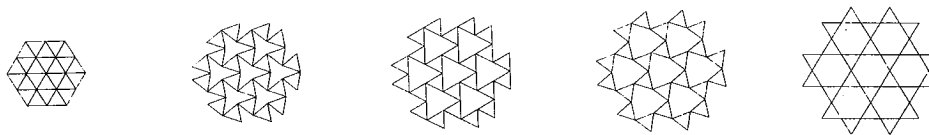
The umbrella concept utilizes rigid component rotation in mutually perpendicular planes about a common point.



5 Variable Geometry Concept:

Figure 2.5

The variable geometry concept utilizes frame arches and a base ring. The arches are attached about the base ring and by means of hinging and actuating, the arches can be rotated to assume the desired configuration. The arches lie in one plane in the compressed state; in the deployed state they are rotated to assume a three-dimensional structural framework.



Tessellation Transformation:



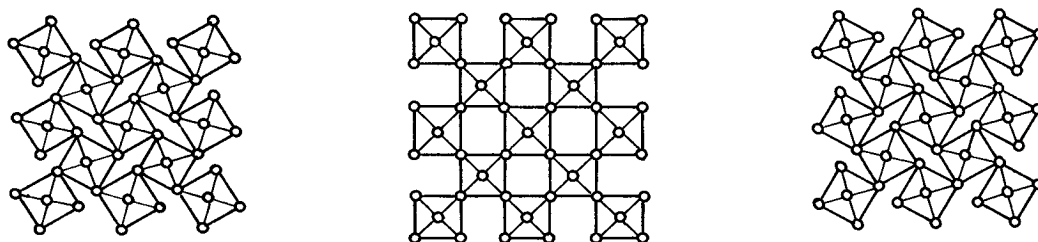
Polyhedral Transformation:

6 Geometric Transformation Concept

Figure 2.6

2.2 TESSELLATION TRANSFORMATION CONCEPT

Crystallographers have long concerned themselves with the study of orderliness occurring in natural crystal structures. One such area is that of "displacive transformation".* Figure 2.7 illustrates this kind of transformation.



Displacive Transformation

Figure 2.7

It is possible in some cases to effect a transformation of the second coordination without having to disrupt the first coordination bond. Thus atoms of one structure may be displaced in such a way as to maintain the same nearest-neighbor relationships while the structure is distorted into a different secondary coordination. Such transformations may therefore be called displacive or distortional transformations. This transformation is especially easy, and indeed quite common, in open structures having at least one low coordination. These structures can be regarded as space networks, and the transformation involves a systematic distribution of the net by bending or straightening the primary bonds without disrupting the linkage of the net.**

* Buerger 1.

** Veroma and Krishna 1.

This section of the report shall discuss the expansion of tessellation forms by means of a transformation similar to the displative transformation. The discussion shall be limited to the regular and semi-regular tessellation forms, however, by no means is the concept of expansion limited to these forms as shall be illustrated later in the report.

A plane or regular tessellation is an infinite set of polygons fitting together to cover the whole plane just once, so that every side of each polygon belongs to one other polygon. It is thus a map with infinitely many faces.* Table 2.1 gives the rules for regular tessellation of which there are only three. Figure 2.8.

The symbols used to describe the tessellation forms (regular and semi-regular) use the modified "Schläfli symbol" which may be read as per example.** 4^4 means there are four squares at each vertex; $3^4.6$ means four triangles and one hexagon at each vertex. Table 2.2.

Table 2.1

Rules For Regular Tessellations

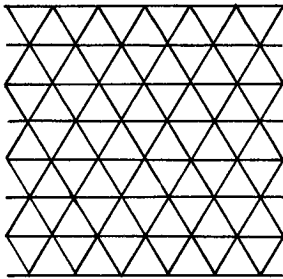
A regular tessellation must contain only one kind of plane polygon, that is equilateral, equiangular, and rectilinear.

Every side of each polygon must belong also to one other polygon with every vertex of each polygon belonging also to one other polygon vertex.

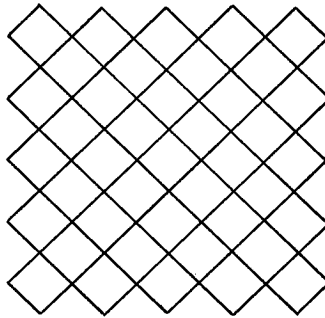
*Coxeter, H. S. M. 1.

**Cundy and Rollett 1.

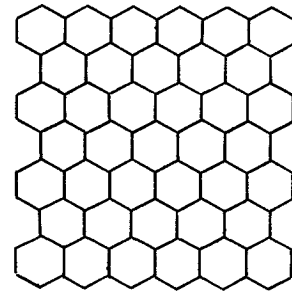
Each tessellation is an infinite set of polygons fitted together to cover the whole plane just once so that a straight line drawn at random on the tessellation will pass the boundaries of any polygon no more than twice.



3^6



4^4



6^3

Regular Tessellations

Figure 2.8

Table 2.2

Notations of the Regular & Semi-Regular Tessellations

Notation (Schläfli Symbol)	No. of polygons around each vertex	Type of Polygon in Tessellation and No. of each type face around vertex				
		3	4	6	8	12
3^6	6	6	-	-	-	-
4^4	4	-	4	-	-	-
6^3	3	-	-	3	-	-
$3^3.4^2$	5	3	2	-	-	-
$3^2.4.4.4$	5	3	2	-	-	-

Table 2.2 (cont.)

3.4.6.4	4	-	2	1	-	-
3.6.3.6	4	2	-	2	-	-
3 ⁴ .6	5	4	-	1	-	-
4.8 ²	3	-	1	-	2	-
3.12 ²	3	1	-	-	-	2
4.6.12	3	-	1	1	-	1

A semi-regular tessellation is a set of regular polygons of two or more kinds so arranged that every vertex is congruent to every other vertex. Table 2.3 gives the rules for the semi-regular tessellation forms of which there are eight. Figure 2.9 Two of these, $3^4.6$ and $3^2.4.3.4$ has two forms which are enantiomorphous; all the others are symmetrical.

Table 2.3

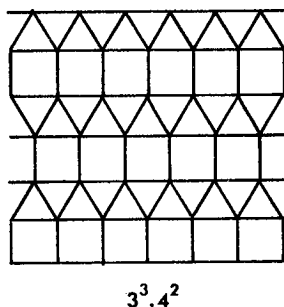
Rules for the Semi-Regular Tessellations

A semi-regular tessellation must contain only plane polygons; more than one kind may be employed in a single map.

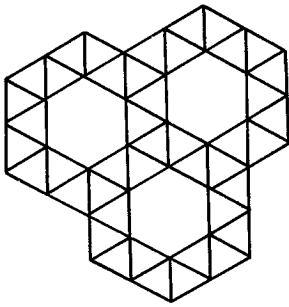
They must also be equilateral, equiangular, and rectilinear.

There must be the same number and kind of polygons joined in the same order (or its enantiomorphic) at each of the vertices of the map with every side of each polygon belonging also to one other polygon.

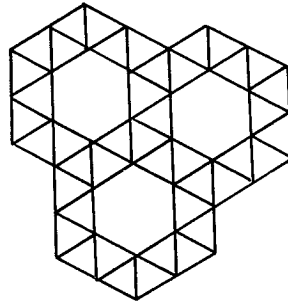
Each tessellation is an infinite set of polygons fitted together to cover the whole plane just once so that a straight line drawn at random on the tessellation will pass the boundaries of any polygon no more than twice.



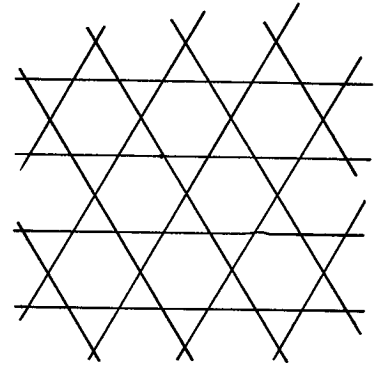
Semi-Regular Tessallations
Figure 2.9



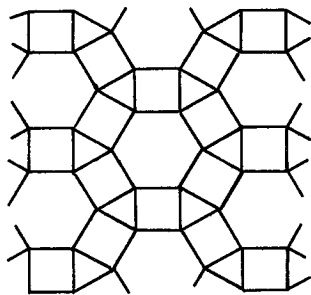
$3^4.6$



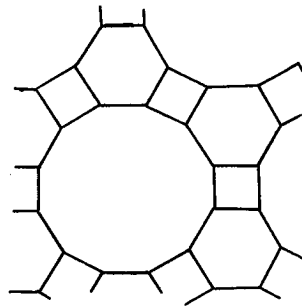
$3^4.6$



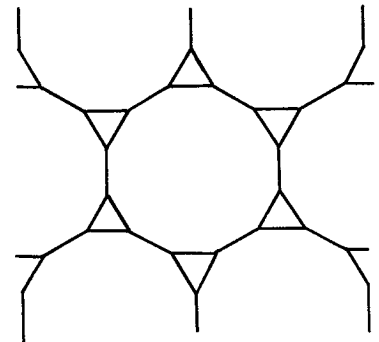
$3.6.3.6$



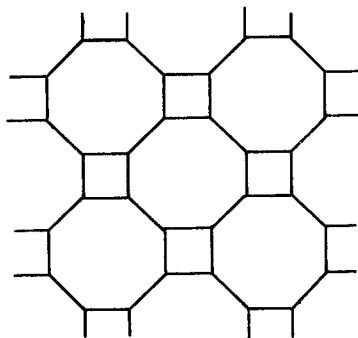
$3.4.6.4$



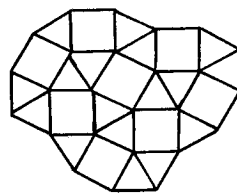
$4.6.12$



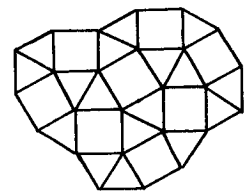
3.12^2



4.8^2



$3^2.4.3.4$



$3^2.4.3.4$

Semi-Regular Tessellations

Figure 2.9 (cont)

Each of these tessellations are involved in transformations similar to the displacive transformation.* The basic geometric expansion incorporates the use of two primitive transformations in a simultaneous operation: Translation (in a certain direction, through a certain distance) and rotation (about a certain axis, through a certain angle.** Figure 2.10

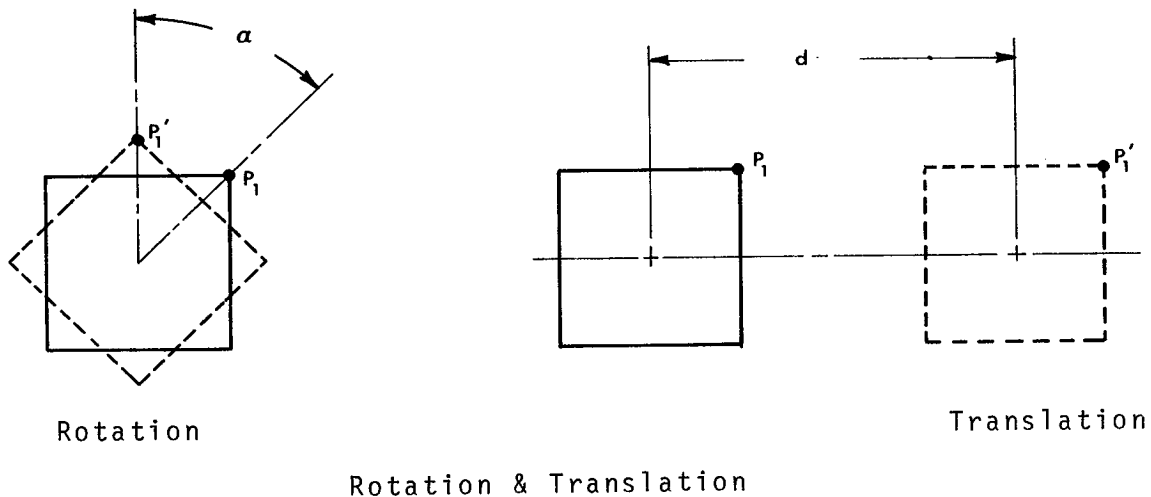
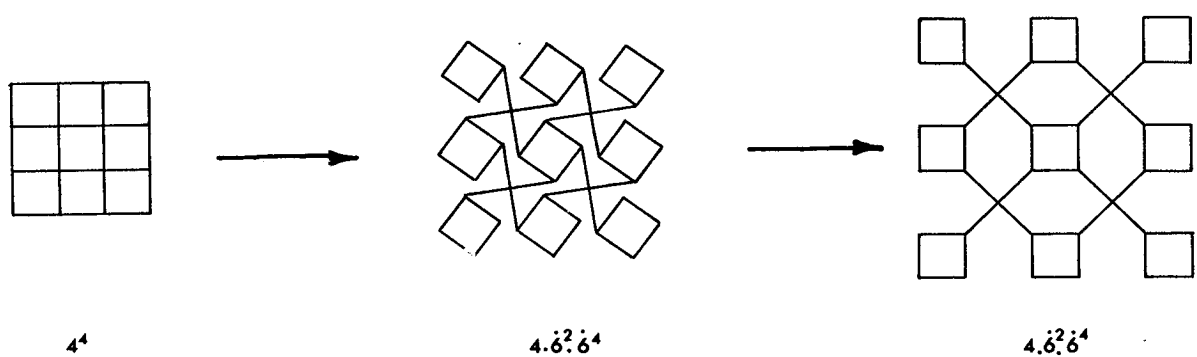
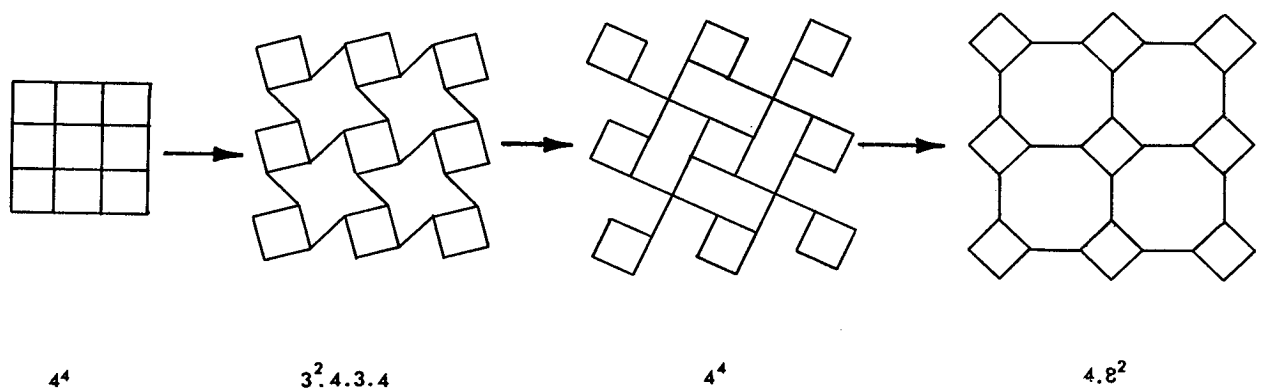
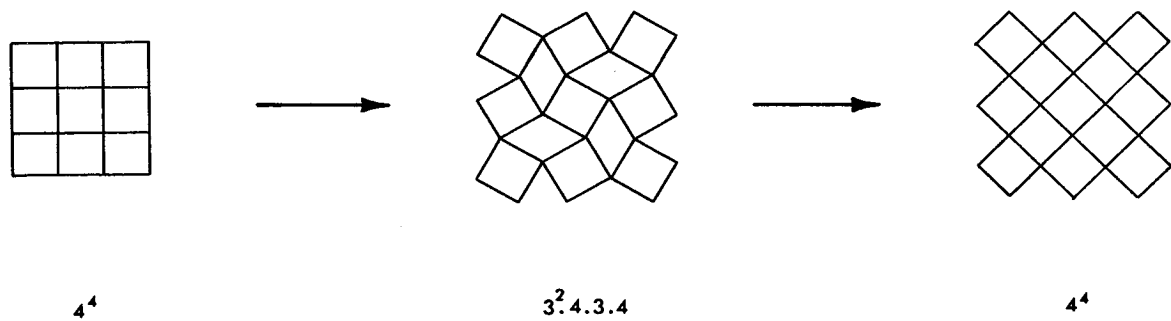


Figure 2.10

There are three basic types of transformations of this nature that may be accomplished with the tessellation nets: face rotation-translation transformation; element rotation-translation transformation; vertice rotation-translation transformation. Figure 2.11.

*There are many other tessellation forms of a higher order than the regular and semi-regular forms that will undergo this type of transformation.

**Medenov and Parkhomenko: 1

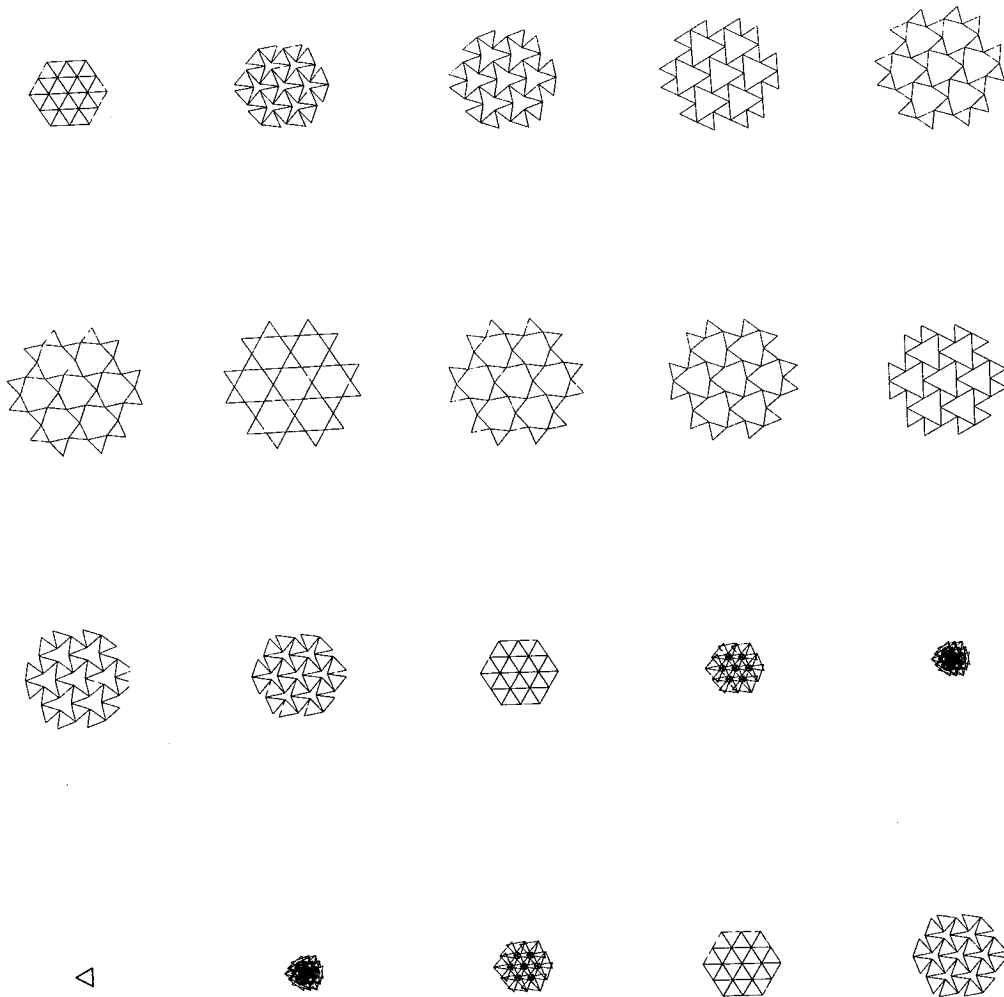


The Three Geometric Transformations
of Regular Tessellation Forms

Figure 2.11

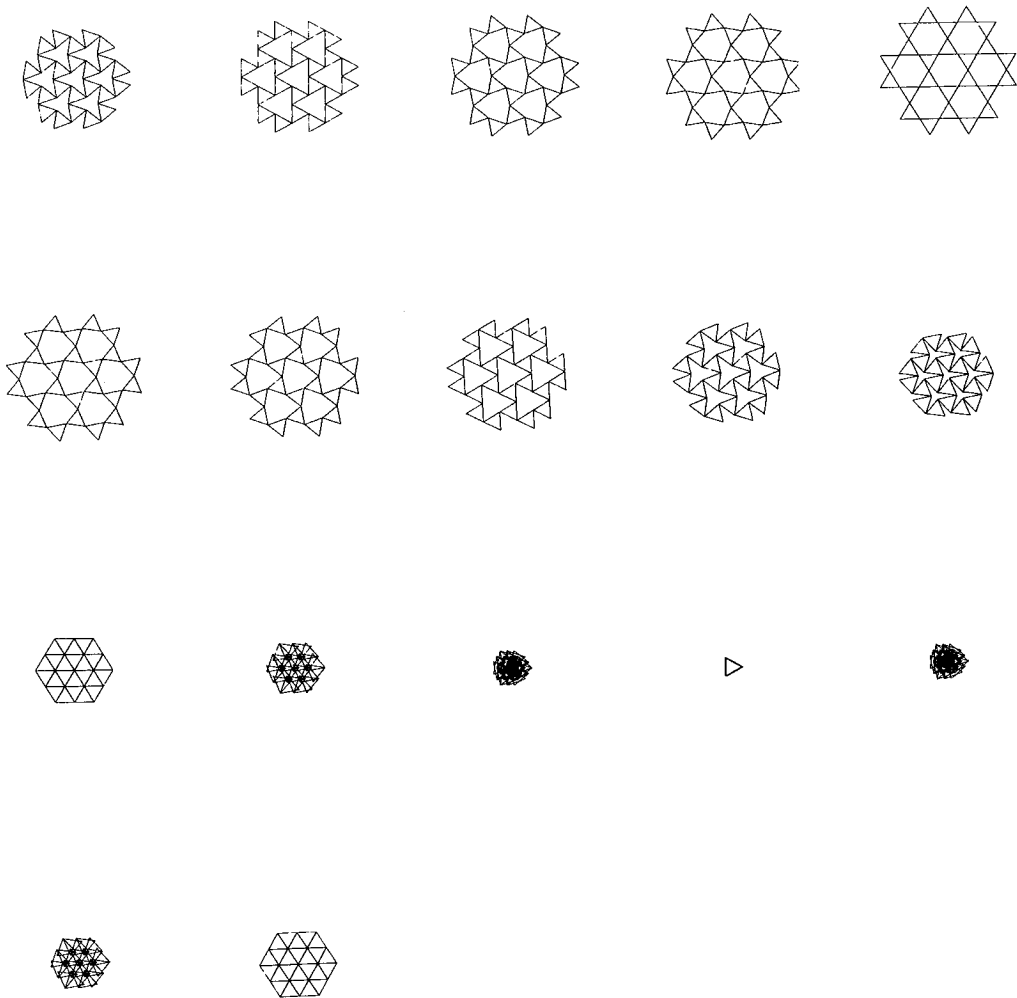
2.3 Transformations of Tessellations

Each of the three forms of the transformations may go through a rotation of 360° in which the tessellation form cycles from a closed state to an open state to an infold state and back to a closed state. Figure 2.12.



360° Transformation of a
3⁶ Tessellation

Figure 2.12



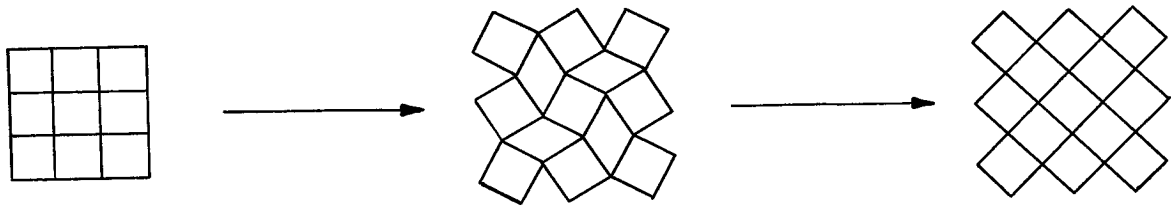
360⁰ Transformation of a
3⁶ Tessellation

Figure 2.12 (cont.)

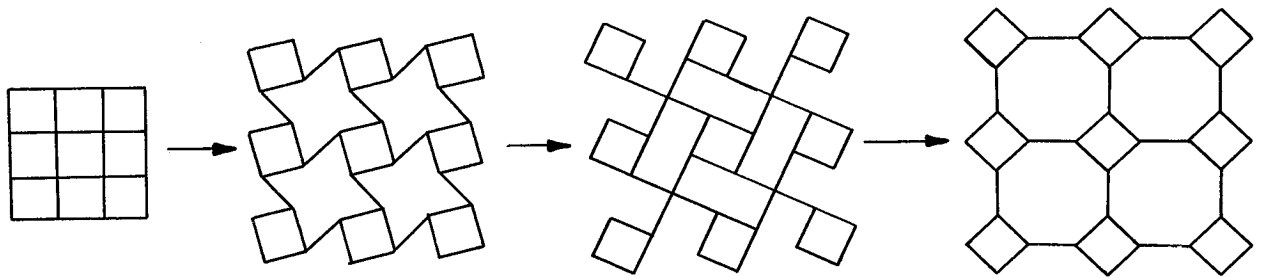
All of the regular and semi-regular tessellation forms and many higher forms have been studied. However, only the regular forms shall be presented here. The following illustrations show the transformation of the three regular tessellation nets using the three methods of expansion. Note the duality existing in the various stages of the transformation. This duality may act as space fillers during various stages of the transformation. Table 2.4, Figure 2.13.

Table 2.4
Transformation Stages

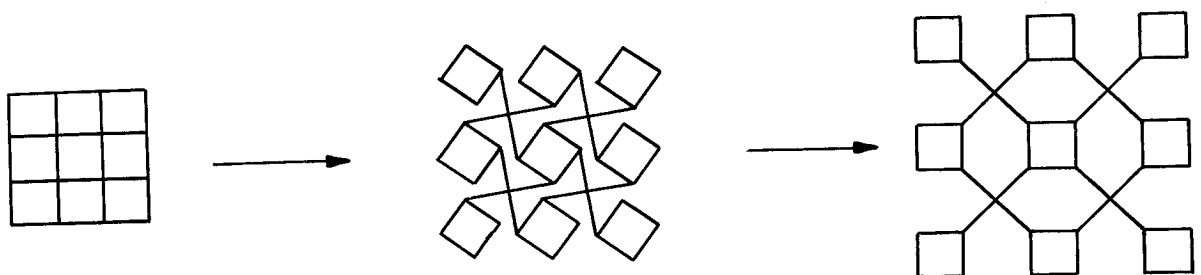
	Closed State	Intermediate State	Open State
1	4^4	$3^2.4.3.4$	4.4
2	4^4	$3^2.4.3.4$	4^4 4.8^2
3	4^4	$4.6^2.6^4$	$4.6^2.6^4$
4	3^6	3^6	$3.6.3.6$
5	3^6	$3^4.6$	3.12^2
6	3^6	3^6	3^6 $3^6.3.4.3^2.4$
7	6^3		$3.6.3.6$
8	6^3	$3^4.6$	6^3
9	6^3	$3^6.6.3^4$	6^3



Face Transformation



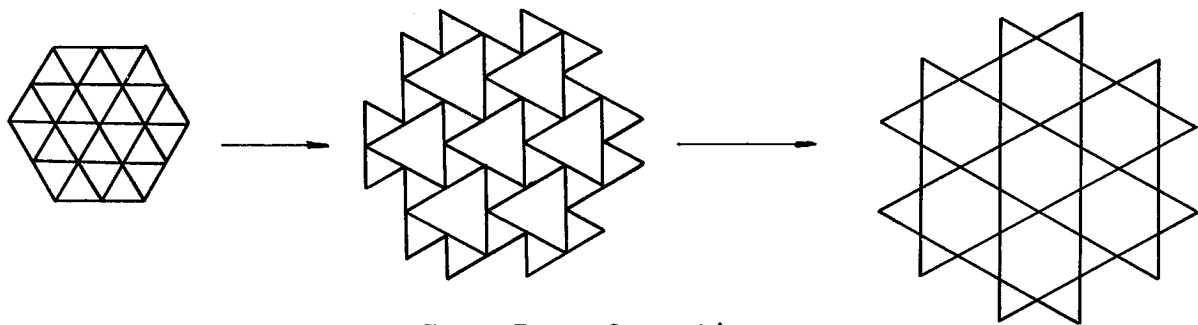
Edge Transformation



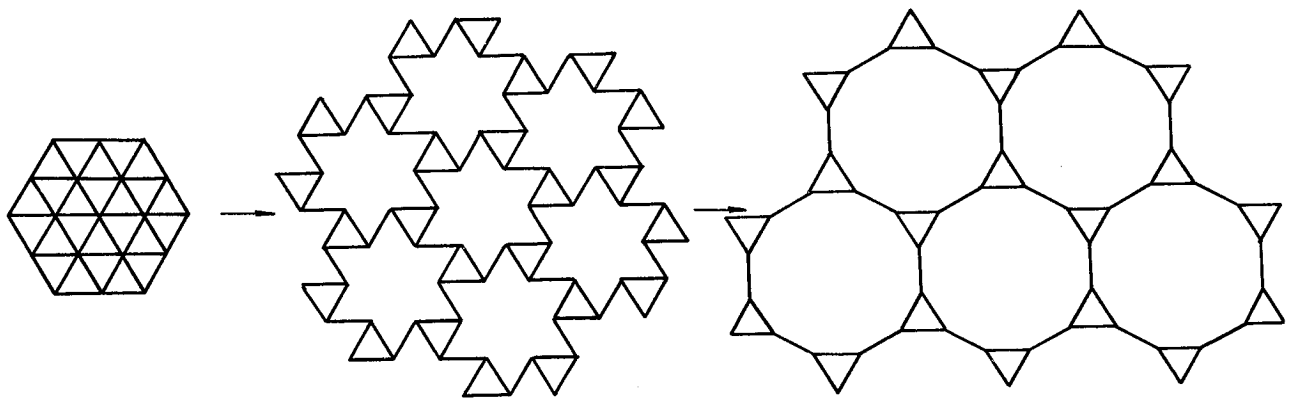
Vertex Transformation

Transformation of Tessellations

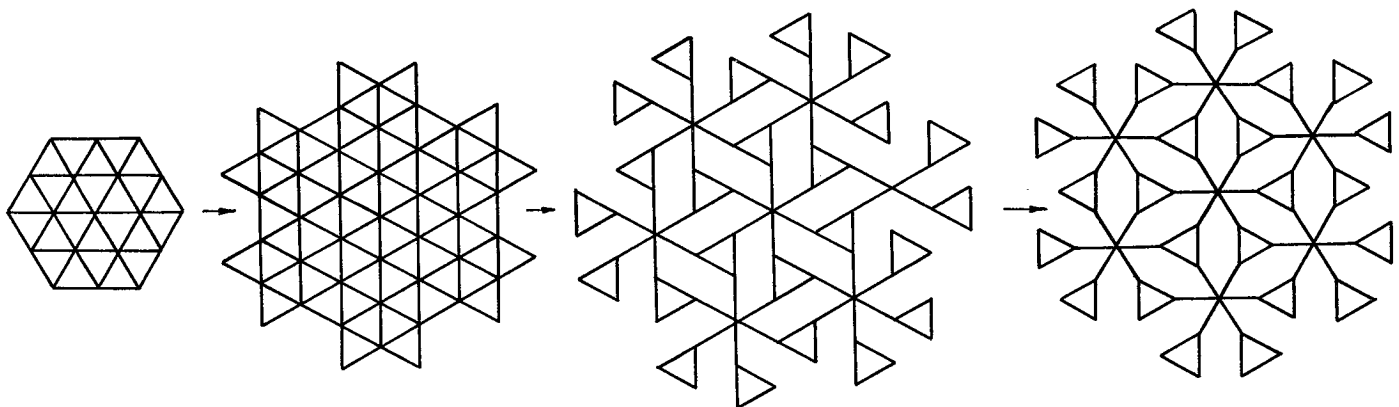
Figure 2.13



Face Transformation



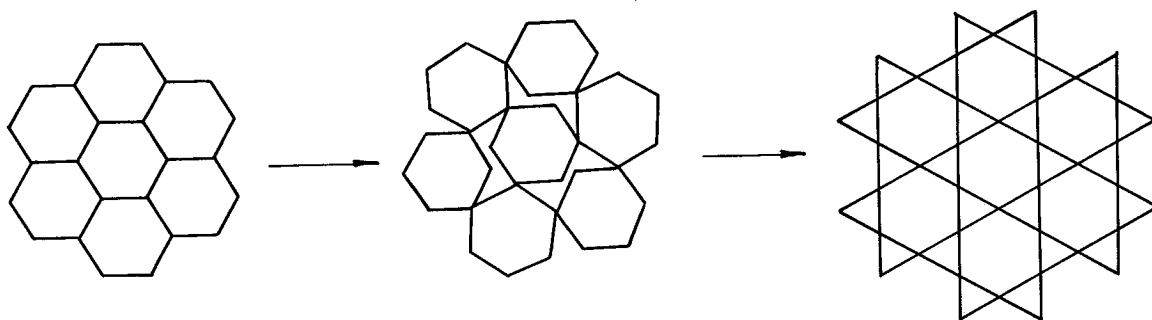
Edge Transformation



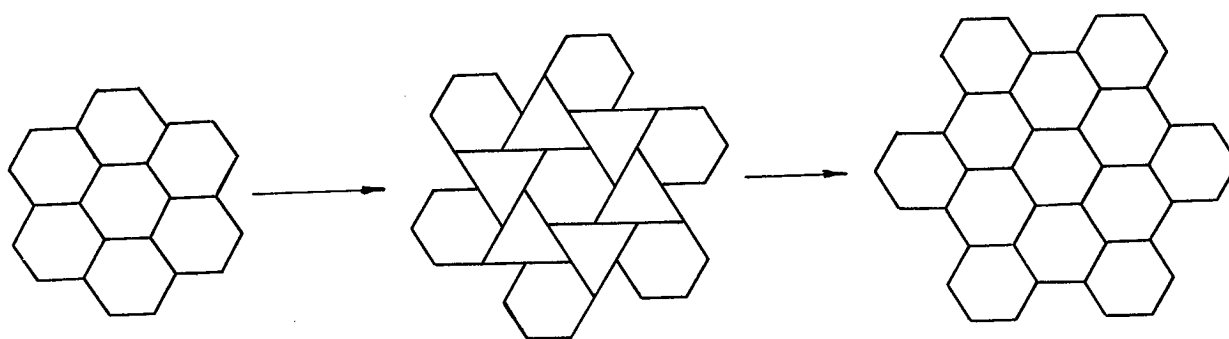
Vertex Transformation

Transformation of Tessellations

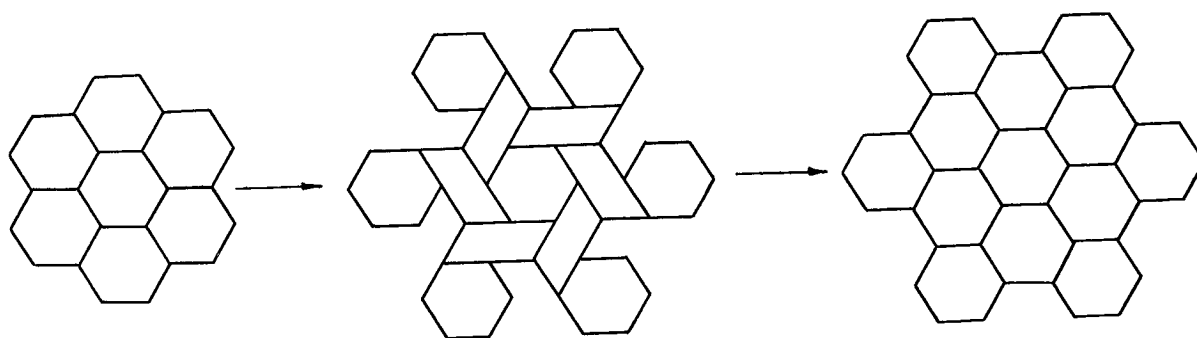
Figure 2.13 (cont.)



Face Transformation



Edge Transformation



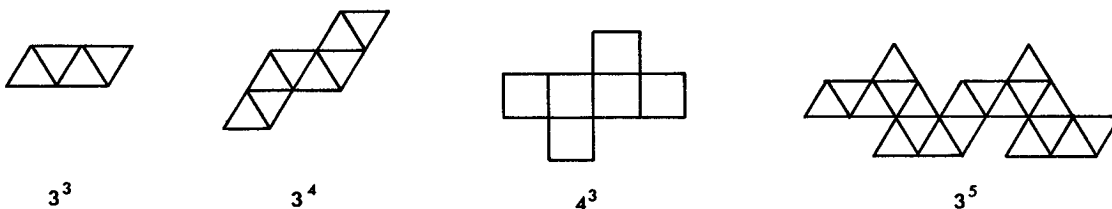
Vertex Transformation

Transformation of Tessellations

Figure 2.13 (cont.)

2.4 TRANSFORMATIONS OF POLYHEDRAL NETS

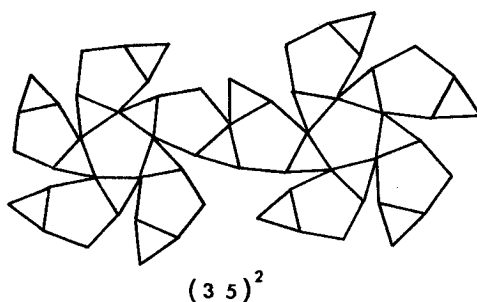
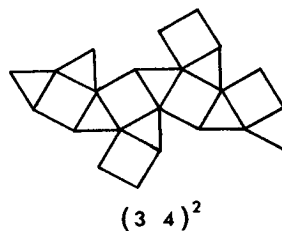
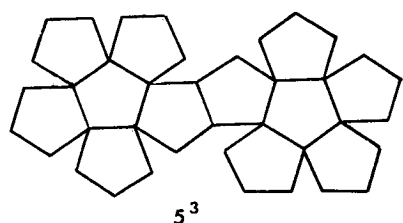
The transformation of polyhedral nets correlates with the investigation of the transformation of polyhedra.* In the investigation centered around the polyhedra, a screw axis transformation existed relating to a three dimensional space, (where there are three directions, x, y, and z.) In the investigation of the polyhedral nets only two dimensions exist (two directions x and y). The polyhedral nets illustrated here are the regular and quasi-regular polyhedral nets. (A net: similar to a tessellation representing the polyhedral faces projected upon a single plane retaining the true size and shape of each face and sharing an edge with an adjacent face.) See Figure 2.14



Regular and Quasi-Regular Polyhedral Nets

Figure 2.14

*The transformation of Polyhedra is discussed later in this part of the report.

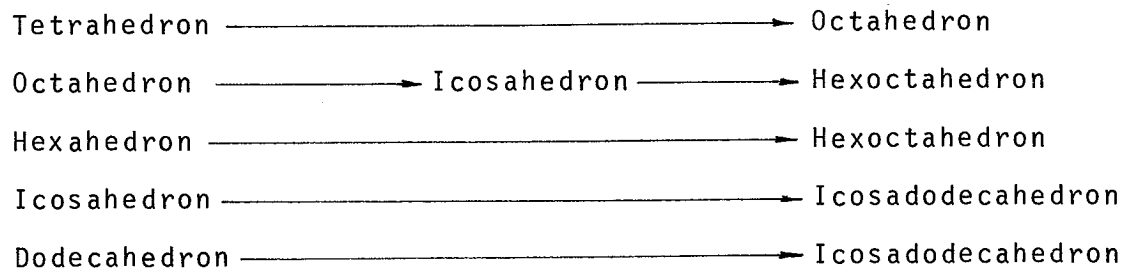


Regular and Quasi-Regular Polyhedral Nets

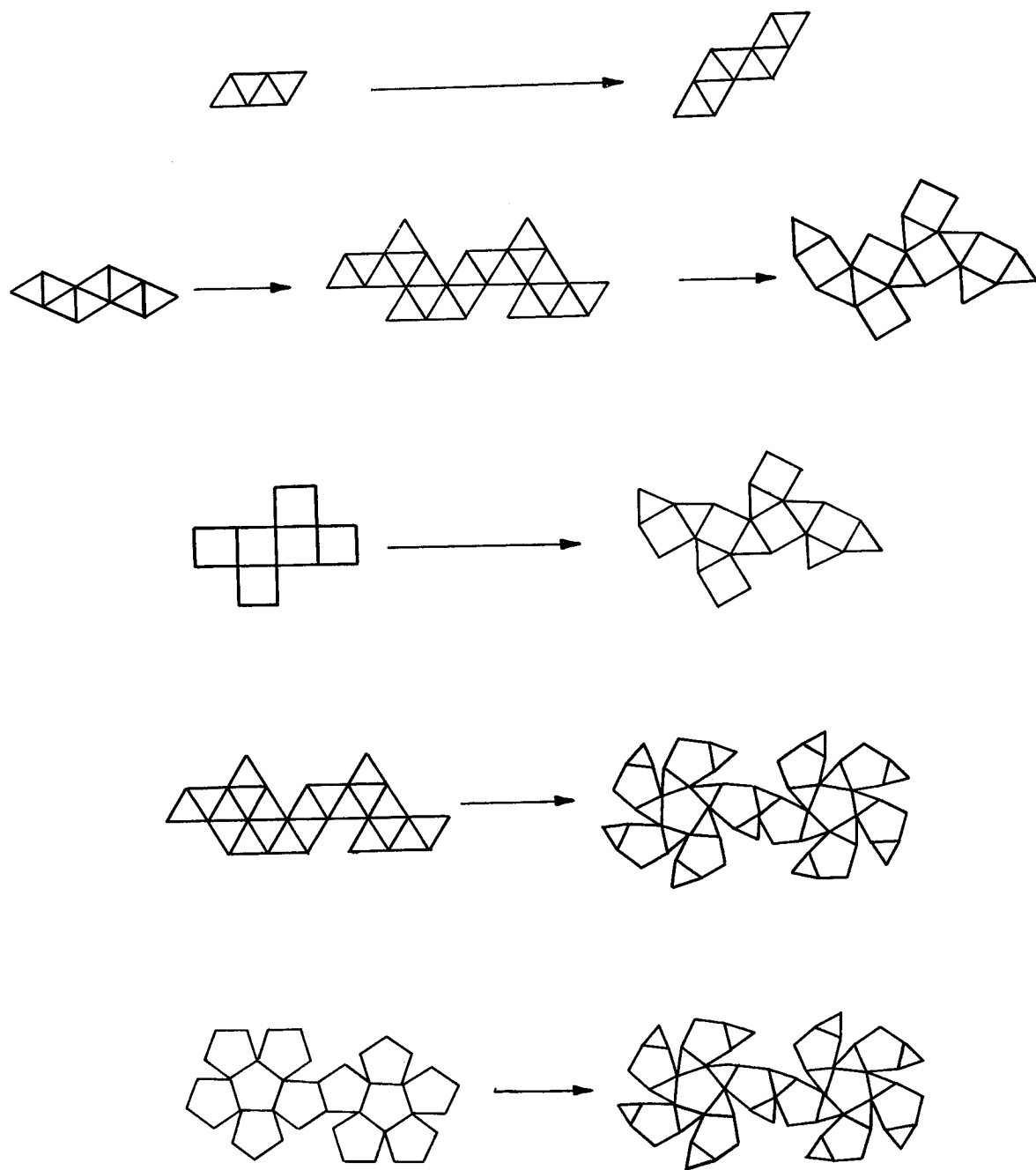
Figure 2.14 (cont)

The polyhedral nets are not necessarily tessellations, however, they will transform in a similar manner. The net transformations are ordered in a direct relationship to the polyhedral transformations. Table 2.5 and Figure 2.15 illustrate the transformations using the face-rotation translation transformation of the regular polyhedral nets. Also illustrated are the intermediate forms created from the transformations which, in many cases, are the semi-regular polyhedral nets.

Table 2.5
Face Transformations of Polyhedra



	<u>F</u>	<u>Schläfli Symbol</u>	<u>3</u>	<u>4</u>	<u>5</u>
Tetrahedron	4	3^3	4	-	-
Octahedron	8	3^4	8	-	-
Hexahedron	6	4^3	-	6	-
Icosahedron	20	3^5	20	-	-
Dodecahedron	12	5^3	-	-	12
Hexoctahedron	14	$(3.4)^2$	8	6	-
Icosadodecahedron	32	$(3.5)^2$	20	-	12



Face Transformation

Figure 2.15

Table 2.6 and Figure 2.16 illustrate the transformations using the edge rotation-translation transformation of the regular polyhedral nets. Also illustrated are the intermediate forms created from the transformations which again, in many cases, form the semi-regular polyhedral nets.

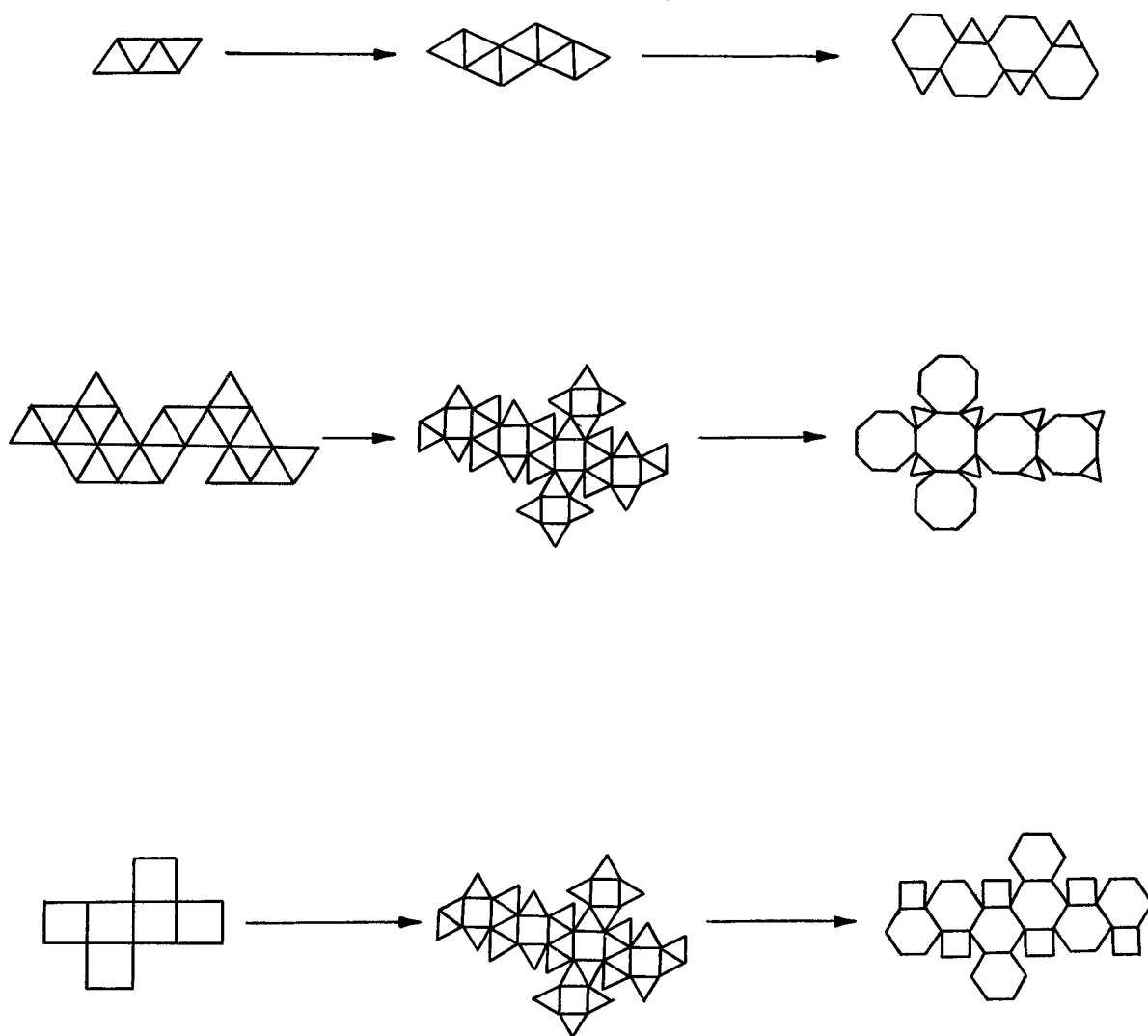
Table 2.6
Edge Transformations of Polyhedral Nets

Tetrahedron \longrightarrow Icosahedron \longrightarrow Truncated Tetrahedron
 Octahedron \longrightarrow Snub Hexahedron \longrightarrow Truncated Hexahedron
 Hexahedron \longrightarrow Snub Hexahedron \longrightarrow Truncated Octahedron
 Icosahedron \longrightarrow Snub Dodecahedron \longrightarrow Truncated Dodecahedron
 Dodecahedron \longrightarrow Snub Dodecahedron \longrightarrow Truncated Icosahedron

	<u>Vertices</u>		<u>Faces</u>	<u>Edges</u>	<u>3</u>	<u>4</u>	<u>5</u>
Tetrahedron	4	3^3	4	6	4	-	-
Octahedron	6	3^4	8	12	8	-	-
Hexahedron	8	4^3	6	12	-	6	-
Icosahedron	12	3^5	20	30	20	-	-
Dodecahedron	20	5^3	12	30	-	-	12
Hexoctahedron	12	$(3.4)^2$	14	24	8	6	-
Icosadodecahedron	30	$(3.5)^2$	32	60	20	-	12

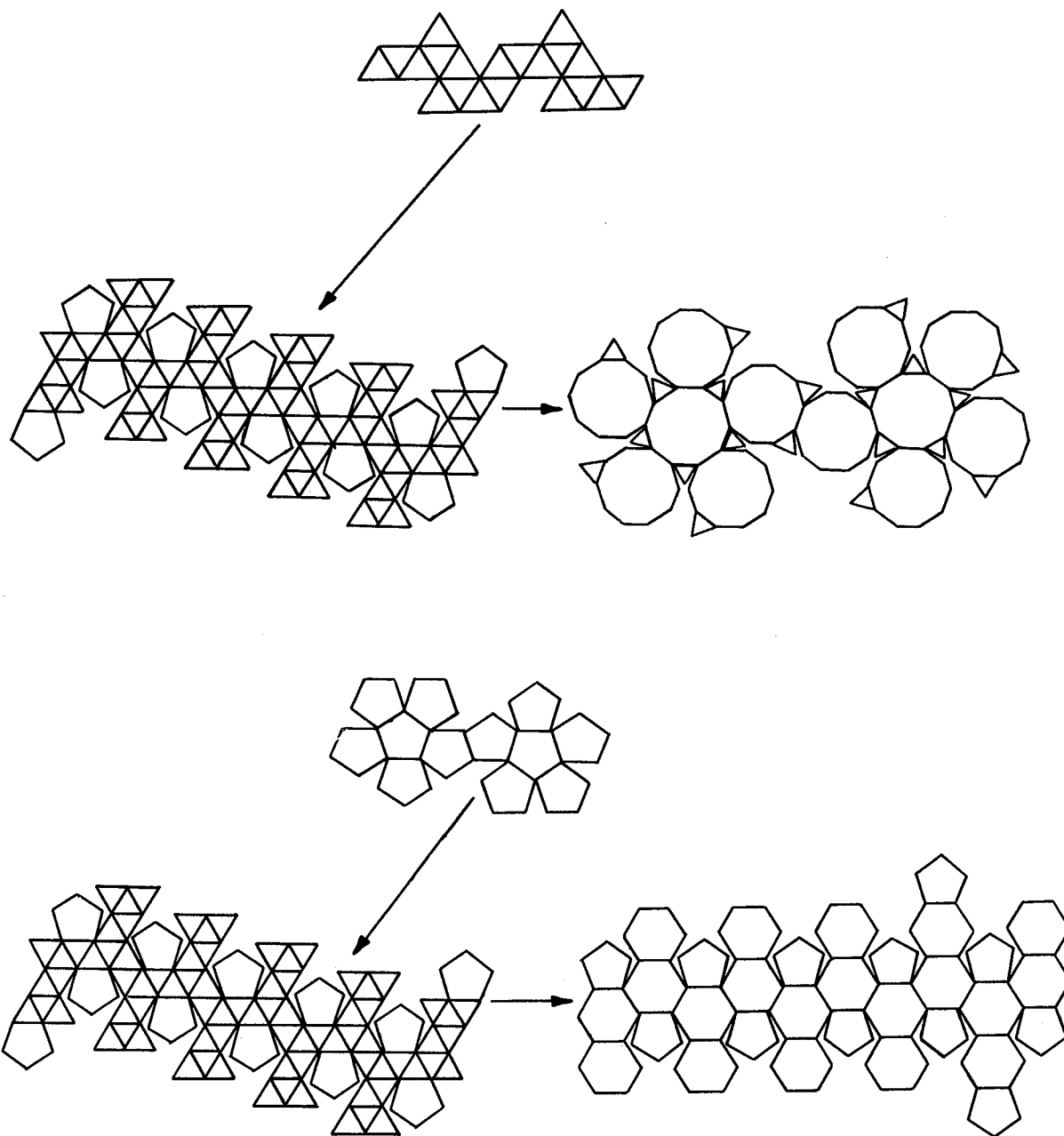
Table 2.6 (cont.)

	<u>Vertices</u>		<u>Face</u>	<u>Edges</u>	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>	<u>8</u>	<u>10</u>
Snub Hexahedron	24	$3^4.4$	38	60	32	-	-	6	-	-
Snub Dodecahedron	60	$3^4.5$	92	150	80	-	12	-	-	-
Truncated Tetrahedron	12	3.6^2	8	18	4	-	-	4	-	-
Truncated Hexahedron	24	3.8^2	14	36	8	-	-	-	6	-
Truncated Octahedron	24	4.6^2	14	36	-	6	-	8	-	-
Truncated Dodecahedron	60	3.10^2	32	90	20	-	-	-	-	12
Truncated Icosahedron	60	5.6^2	32	90	-	-	12	-	20	-



Edge Transformation of Polyhedral Nets

Figure 2.16



Edge Transformation of Polyhedral Nets

Figure 2.16 (cont.)

2.5 Computer Model of Tessellation Transformation

The computer program and plot routine was written for the IBM 7040/7044 computer, utilizing FORTRAN IV language. The program may be used for a 3^6 , 4^4 , or 6^3 tessellation depending upon the input data. The program only considers a face rotation-translation transformation with the output in the form of time duration maps based on a constant rate of rotation.

Example of input data is given in Table 2.7. The examples of output maps are oriented with a common vertex as the center (0,0) around which the rotation and translation is taken. Figures 2.17-2.22.

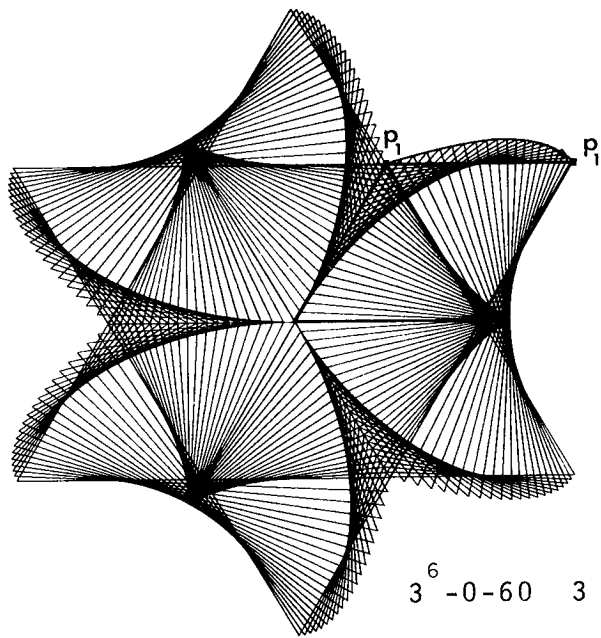
Table 2.7

Sample Data Input Cards For TESTSH

<u>Format</u>	<u>Columns</u>	<u>Description</u>
I3	1-3	Minimum tessellation angle in degrees.
I3	4-6	Maximum tessellation angle in degrees.
I3	7-9	Increment from minimum to maximum.
I1	10	No. of runs in tessellation (max. to be 3)
F10.0	11-20	Length of members (assumed 1 in)
A6	21-26	if Move <u> </u> is placed here, every angle will be plotted separately

Table 2.7 (cont.)
Sample Input for 3⁶

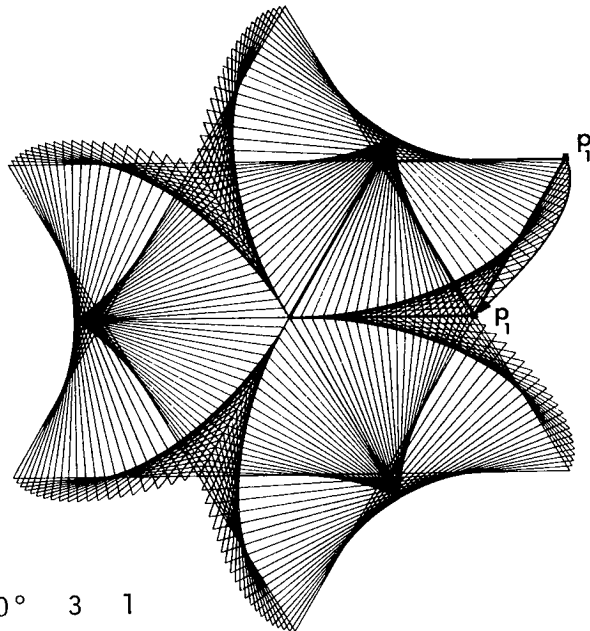
<u>Format</u>	<u>Columns</u>	<u>Description</u>
A6	27-32	Type of tessellation (TRI ₁₁₁ -triangle SQUA ₁₁ Square, HEX ₁₁₁ - hexagon)
3X	33-35	Blank
F10.0	36-45	Rotation of axis clock- wise (in degrees)
I3	46-48	Type of Plot (1-polygon, 2-path or vertex, 3- diagonal of polygon, square only)
I3	49-51	Blank



$3^6 - 0 - 60 \quad 3 \quad 1$

Figure 2.17

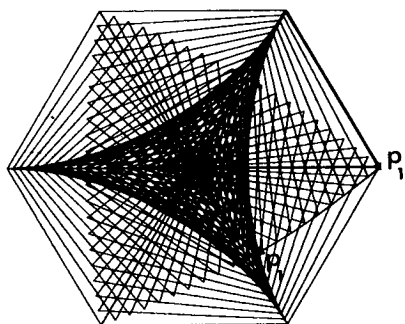
The time duration map illustrated is from a closed position to an₆ open position $0^\circ - 60^\circ$ at a 3° rotation interval of a one row 3^6 tessellation.



$3^6 - 60 - 120^\circ \quad 3 \quad 1$

Figure 2.17 (cont.)

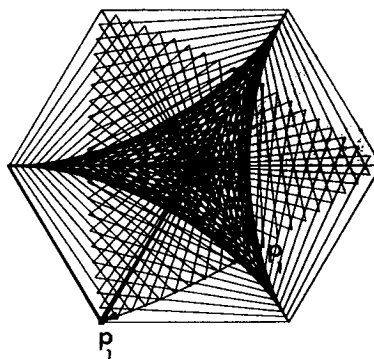
This continuation of the previous map illustrates the next phase of the cycle where P_1 is in the open position at 60° and returns to the closed position at 120° with a 3° rotation interval of a one row 3^6 tessellation



3^6 120-150 3 1

Figure 2.17 (cont.)

This continuation of the previous map illustrates the next position of the cycle where P_1 is in the closed position at 120° and is at the IN position at 150° with a 3° rotation interval of a one row 3^6 tessellation.

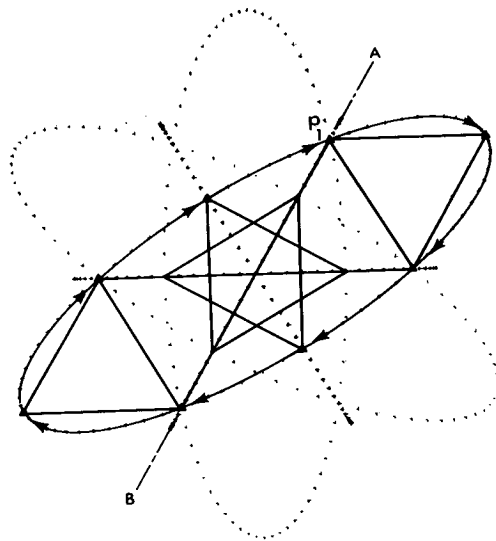


3^6 150-180 3 1

Figure 2.17 (cont.)

The final continuation of the transformation illustrates the next position of the cycle where P_1 is in the IN position of 150° and is at the closed position at 180° with a 3° rotation interval of a one row 3^6 tessellation.

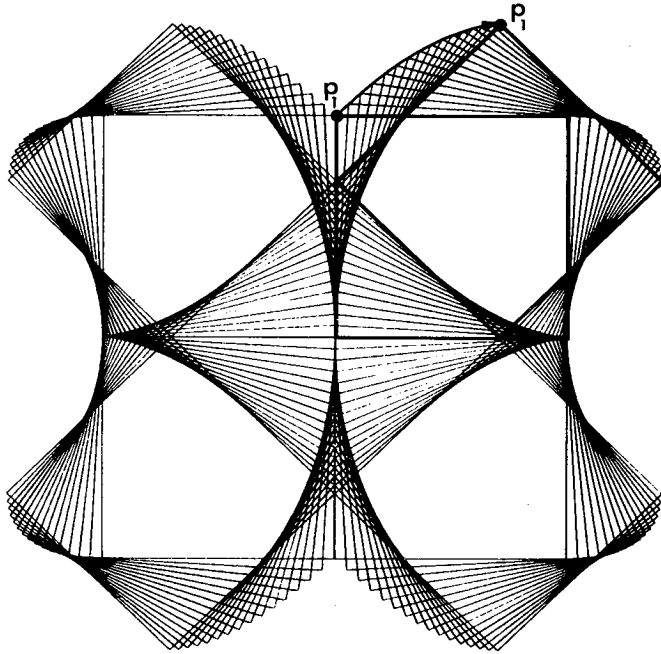
The remaining positions of the total 360° cycle is illustrated through a rotation of the $0-180^\circ$ position of the cycle about the AB axis shown in Figure 2.18.



3⁶ 0-360 5 1

Figure 2.18

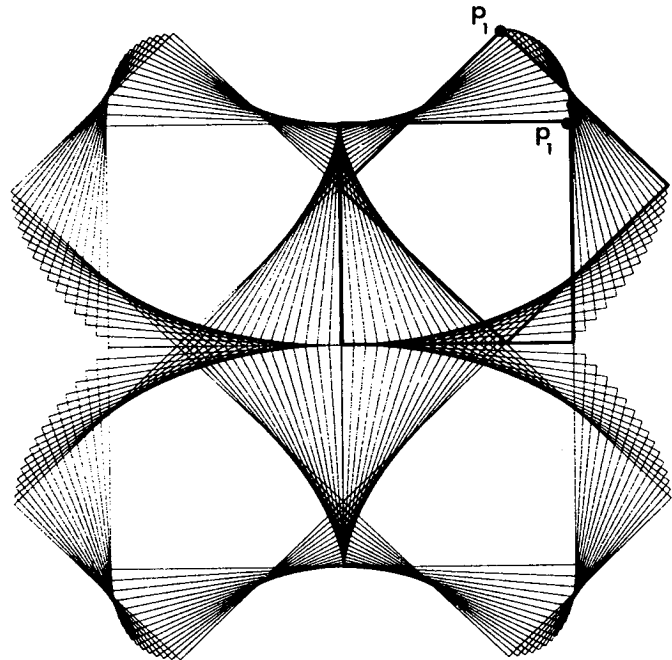
The path P_1 takes during a 360° transformation is illustrated through its entire cycle. Any given vertex will take the form of an ellipse; under special conditions the ellipse may take the form of a circle or a straight line.



4⁴ 0-45 3 1

Figure 2.19

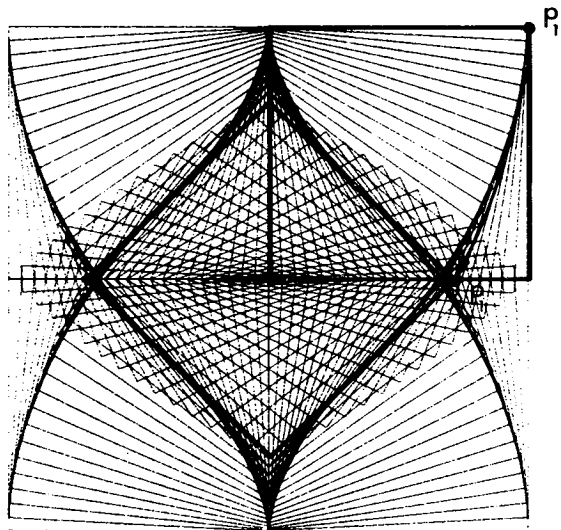
The time duration map illustrated is from a closed position to an open position 0°-45° at 3° rotation intervals of a one row 4⁴ tessellation.



4⁴ 45-90 3 1

Figure 2.19 (cont.)

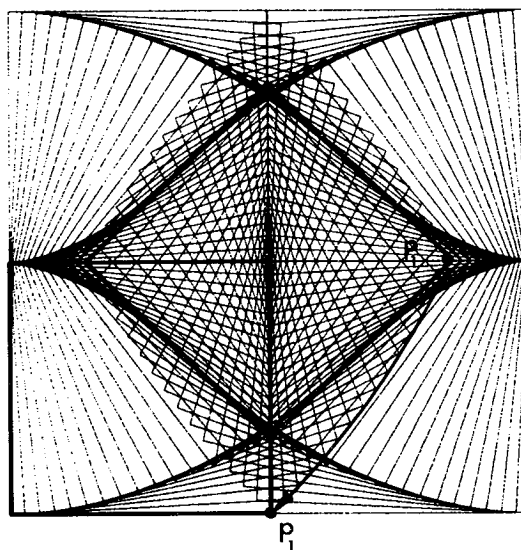
This continuation of the previous map illustrates the next phase of the cycle where P₁ is in the open position at 45° and returns to a closed position at 90° with a 3° rotation interval of a one row 4⁴ tessellation.



4⁴ 90-135 3 1

Figure 2.19 (cont.)

This combination of the previous map illustrates the next portion of the cycle where P1 is in the closed position at 90° and is in the IN position at 135° with a 3° rotation interval of a one row 4⁴ tessellation.

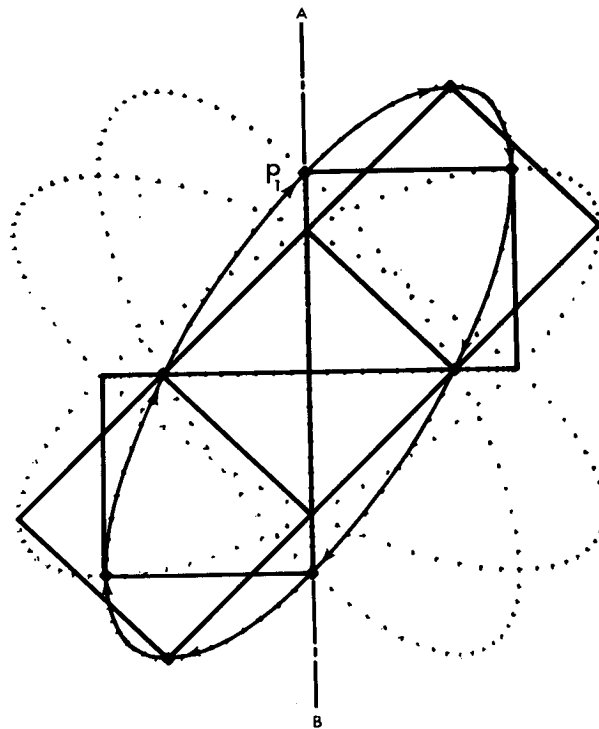


4⁴ 135-180 3 1

Figure 2.19 (cont.)

The final continuation of the transformation illustrates the next portion of the cycle where P1 is in the IN position at 135° and is at the closed position at 180° with a 3° rotation interval of a one row 4⁴ tessellation.

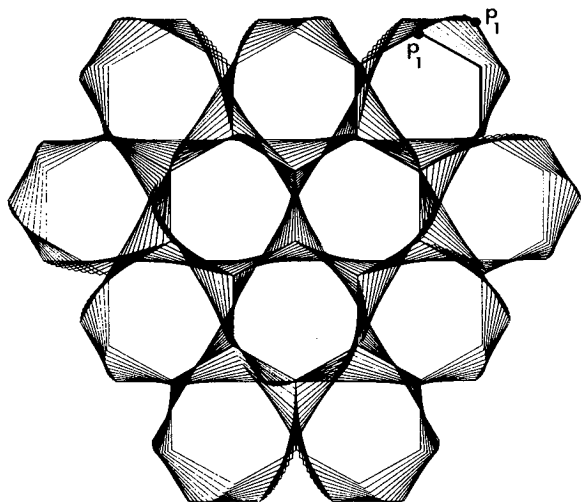
The remaining portions of the total 360° cycle is illustrated through a rotation of the $0-180^\circ$ portion of the cycle about the axis AB shown in Figure 2.20.



44 0-360 5 1

Figure 2.20

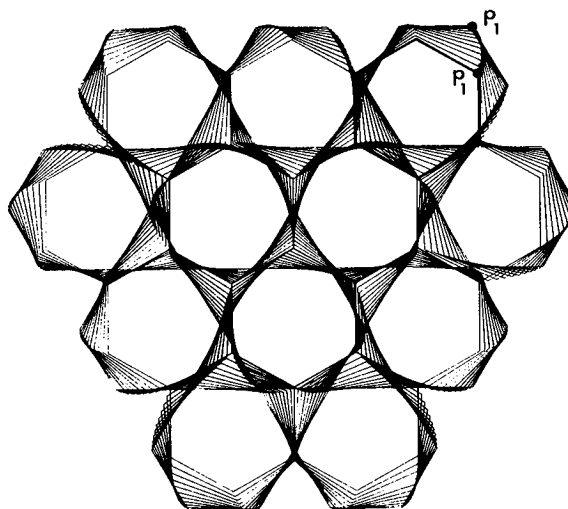
The path P_1 takes during a 360° transformation is illustrated through its entire cycle. Any given vertex will take the form of an ellipse, under special conditions the ellipse may take the form of a circle or a straight line.



6^3 0-30 3 2

Figure 2.21

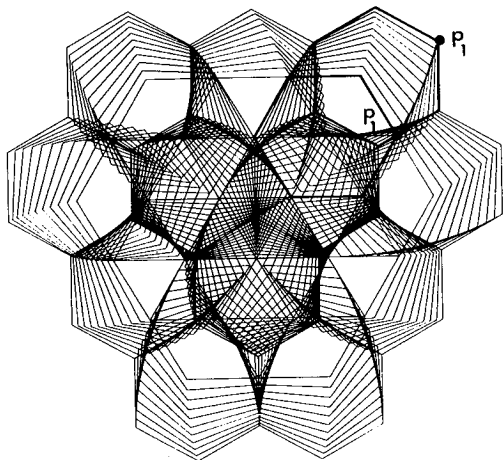
The time duration map illustrated is from a closed position to an open position 0-30 at 3° rotation intervals with P_1 on a hexagon in the second row of a 6^3 tessellation.



6^3 30-60 3 2

Figure 2.21 (cont.)

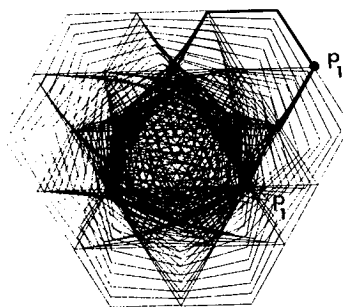
This continuation of the previous map illustrates the next phase of the cycle where P_1 is in the open position at 30° and returns to a closed position at 60° with a 3° rotation interval of a two row 6^3 tessellation.



6^3 60-90 3 2

Figure 2.21 (cont.)

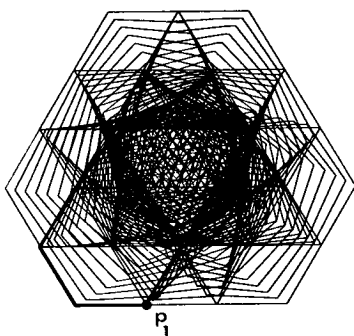
This continuation of the previous map illustrates an intermediate position of the cycle from 60° - 90° . The rotation interval is 3° with a 6^3 tessellation of 2 rows.



6^3 90-120 3 2

Figure 2.21 (cont.)

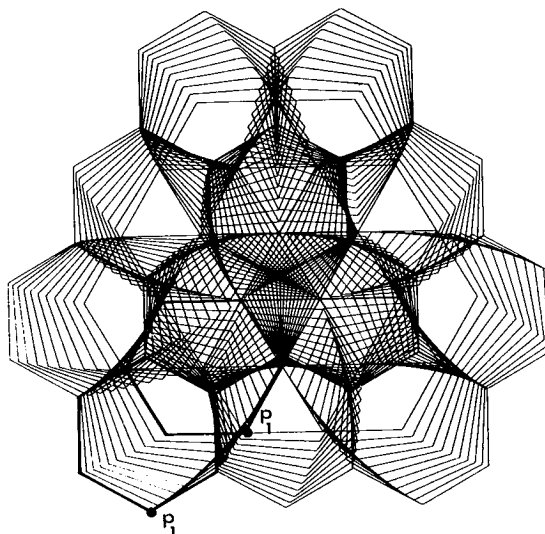
This continuation of the previous map illustrates the IN position of the transformation of P_{13} when it reaches 120° . The rotation interval is 3° with a 6^3 tessellation of 2 rows.



6^3 120-150 3 2

Figure 2.21 (cont.)

This continuation of the previous map illustrates an intermediate position of the transformation of P_1 at a rotation of 150° . The rotation interval is 3° with a 6^3 tessellation of 2 rows.



6^3 150-180 3 2

Figure 2.21 (cont.)

This final continuation of the transformation illustrates the closed position of P_1 at a rotation of 180° . The rotation interval is 3° with a 6^3 tessellation of 2 rows.

THE COMPUTER PROGRAM DESCRIBED
ON PAGES II-40 to II-65
IS AVAILABLE FROM COSMIC

2.6 Polyhedral Transformation Concept

The concept of polyhedral transformation can be said to have begun with R. Buckminster Fuller's discussion of his Energetic-Synergetic Geometry.* If spheres are closely packed around a central sphere a polyhedron bounded by fourteen faces is formed. This polyhedron consists of six squares and eight triangles; Fuller describes this polyhedron as a Vector Equilibrium.

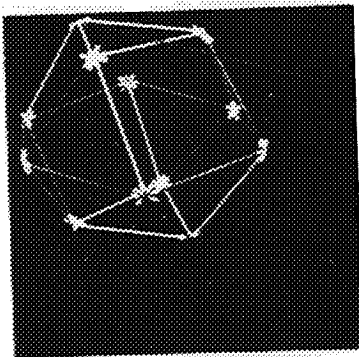
The polyhedron, commonly known as the *Hexoctahedron* or *Cuboctahedron*, is literally an equilibrium of vectors. The value of its radial vectors is exactly the same as that of its circumferential vectors.** The length of the distance from any of the polyhedron's center to its vertices is equal to the length of any of its elements. For this reason an equilibrium exists where the lines of force radiate from its center, and bind inward around its periphery.

Fuller indicated that the removal of the center sphere would cause a significant change in the close packing of spheres, a 20-sided polyhedron would result-- an *icosahedron*. This change suggested that a Vector Equilibrium could be transformed into an icosahedron and vice versa. The same number of surface-defining spheres exist between the two polyhedra and they both have 12 vertices. Each has symmetrical similarities. A family of relationships which is

*Fuller, R. B., 1

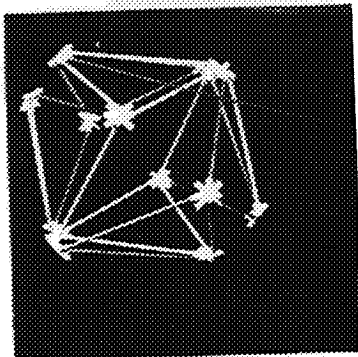
**Marks, R. W., 1

capable of cycling through a sequence of phases existed which Fuller called "*Regenerative*". This sequence of phases can be visually illustrated with the construction of the "*Jitterbug*". The sequence starts with the Vector Equilibrium and when released it compresses symmetrically into an icosahedron and then into an octahedron. Figure 2.23.



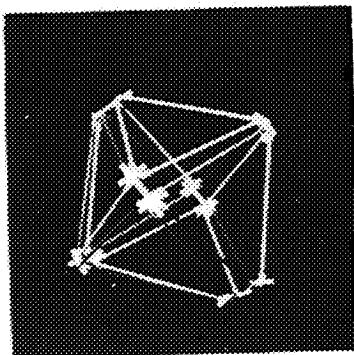
The Vector Equilibrium Phase

The Vector Equilibrium is constructed with circumferential vectors only and with flexible joints.



The Icosahedron Phase

As the top triangle is lowered toward the opposite triangles, rotation moves the vertices into the Icosahedron phase. NOTE: That each of the pairs of opposite triangles, although in motion, maintain an axial relationship.

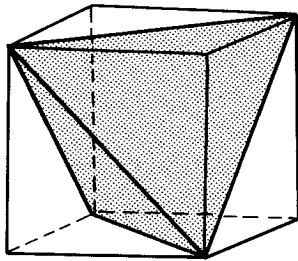


The Octahedron Phase

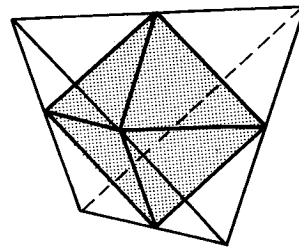
The completion of the cycling results in the phase of an Octahedron.

Figure 2.23

Investigation of the "Regenerative" principal coupled with investigation of the various convex polyhedron*, lead to the concept of polyhedral expansion by rotation-translation methods.** This concept of polyhedral transformation incorporates the use of the familiar regular and semi-regular polyhedron.*** It introduces a new dimension to the old "classical" concepts of polyhedra transformation (Figure 2.24) by introducing a new concept of Rotation-Translations Transformation of polyhedra.



Hexahedron-Tetrahedron



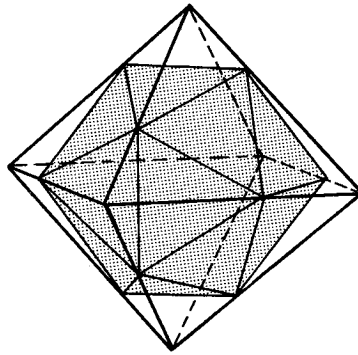
Tetrahedron-Octahedron

Figure 2.24

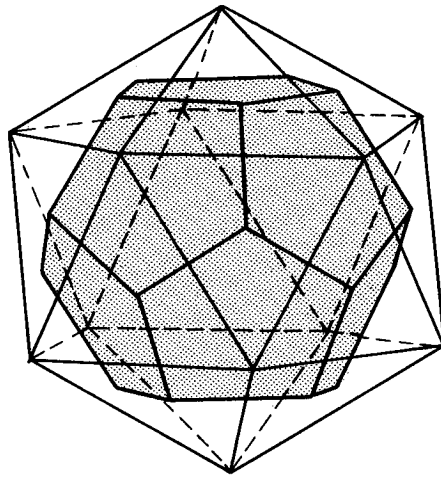
*A convex polyhedron is a polyhedron which has no entrant edges.

**An interview held with R. Buckminster Fuller on May 3, 1965.

*** Stuart P. 1



Octahedron-Icosahedron



Icosahedron-Dodecahedron

Figure 2.24 (cont)

Classical Transformation of Platonic Solids

The Hexahedron is converted into a Tetrahedron by an alternate removal of vertices. The Tetrahedron is converted into an Octahedron by truncation of its vertices. The Octahedron is transformed into the Icosahedron by a somewhat more complex, truncation, and similarly the Icosahedron is converted into the Dodecahedron. A prior knowledge of the forms is necessary before the complete series of transformation may be accomplished.

In contrast to the "classical" method of transformation of polyhedra this new concept maintains dimensionality during transformation and generates a polyhedral form without recourse to any special knowledge other than the rules of transformation described in Table 2.8.

This concept of Rotation-Translation Transformation is characteristic of all regular and semi-regular polyhedra (Table 2.9). By allowing each surface to rotate about its axis, translate along its axis, and maintain connection with one of its paired vertices; the surfaces enclosing the polyhedron will transform into another polyhedral form.*

*Stuart D., 1

Table 2.8

DEFINING RULES FOR REGULAR AND SEMI-REGULAR POLYHEDRA*

1. A Regular Polyhedron must enclose a volume of space with a surface composed of only one kind of plane polygon.
2. These plane polygons must be equilateral, equiangular, and rectilinear.
3. The polygons must mutually join at their edges and vertices so as to completely fill a single imaginary spherical surface passing through the joined vertices.
4. In a similar fashion, a Semi-regular Polyhedron must be totally composed of plane polygons as defined in rule 2, but now, more than one kind of polygon may be used in a single polyhedron.
5. There must be the same numbers and kinds of polygons, joined in the same order (or its enantiomorph) at each of the vertices of the polyhedral surface.
6. For Regular and Semi-regular Polyhedra, the corner angles which join at a single vertex must total in aggregate, less than 360° .
7. The plane of any polygon, if extended, must not pass through the interior volume of the polyhedron. And, a plane passed through the polyhedron at random will always have a single closed polygon at its line of intersection with the polyhedral surface.

*Stuart 1., p. 5.

Table 2.9
CHARACTERISTICS OF THE
REGULAR AND SEMI-REGULAR POLYHEDRA

Name	Equilateral triangle	Square	Pentagons	Hexagons	Octagons	Decagon	Designation		Faces	Edges	Vertices
							Schläfli	Symbol*			
Tetrahedron	4	-	-	-	-	-	3^3		4	6	4
Hexahedron	-	6	-	-	-	-	4^3		6	12	8
Octahedron	8	-	-	-	-	-	3^4		8	12	6
Dodecahedron	-	-	12	-	-	-	5^3		12	30	20
Icosahedron	20	-	-	-	-	-	3^5		20	30	12
Truncated Tetrahedron	4	-	-	4	-	-	$3 . 6^2$		8	18	12
Hexoctahedron	8	6	-	-	-	-	$(3 . 4)^2$		14	24	12
Truncated Hexahedron	8	-	-	-	6	-	$3 . 8^2$		14	36	24
Truncated Octahedron	-	6	-	8	-	-	$4 . 6^2$		14	36	24
Small rhomibu- boctahedron	8	18	-	-	-	-	$3 . 4^3$		26	48	24
Great rhomibu- boctahedron	-	12	-	8	8	-	$4 . 6 . 8$		26	72	48
Snub Hexa- hedron	32	6	-	-	-	-	$3^4 . 4$		38	60	24
Icosadodeca- hedron	20	-	12	-	-	-	$(3 . 5)^2$		32	60	30

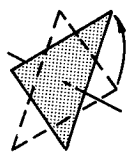
Table 2.9 (cont.)

CHARACTERISTICS OF THE
REGULAR AND SEMI-REGULAR POLYHEDRA

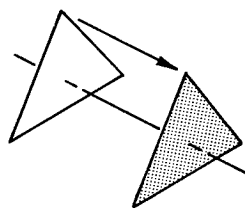
Name	Equilateral triangle	Designation					Schläfli Symbol*	Faces	Edges	Vertices
		Square	Pentagons	Hexagons	Octagons	Decagon				
Truncated dodecahedron	20	-	-	-	-	12	$3 \cdot 10^2$	32	90	60
Truncated Icosahedron	-	-	12	20	20	-	$5 \cdot 6^2$	32	90	60
Small rhombicosi-dodecahedron	20	30	12	-	-	-	$3 \cdot 4 \cdot 5 \cdot 4$	62	120	60
Great rhombicosi-dodecahedron	-	30	-	20	20	12	$4 \cdot 6 \cdot 10$	62	180	120
Snub dodecahedron	80	-	12	-	-	-	$3^4 \cdot 5$	92	150	60

*Kravitz 1., p. 119.

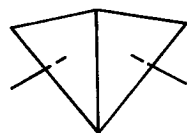
Each of these polyhedra are involved in transformations similar to that of the "Jitterbug" transformation. The basic transformation incorporates the use of the same two primitive transformations discussed in the section on tessellations: Translation (in a certain direction, through a certain distance) and, Rotation (about a certain axis, through a certain angle). Crystallographers refer to this transformation as the screw axis transformation. There are again three basic types of transformation that may be accomplished with the polyhedral forms: By face rotation-translation transformation, element rotation-translation transformation, and vertice rotation-translation transformation. Figure 2.25



Rotation



Translation



Rotation-Translation

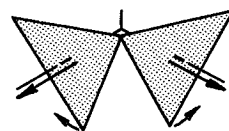
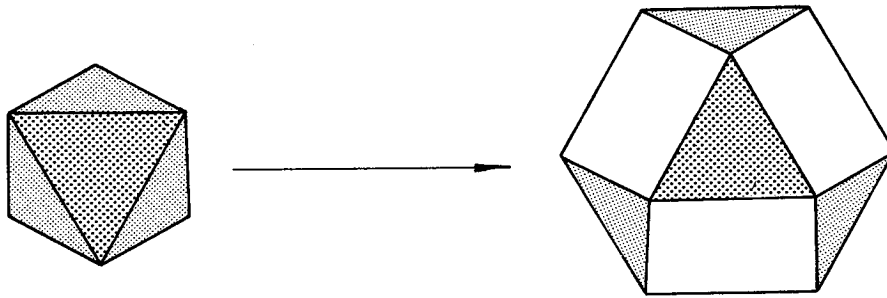
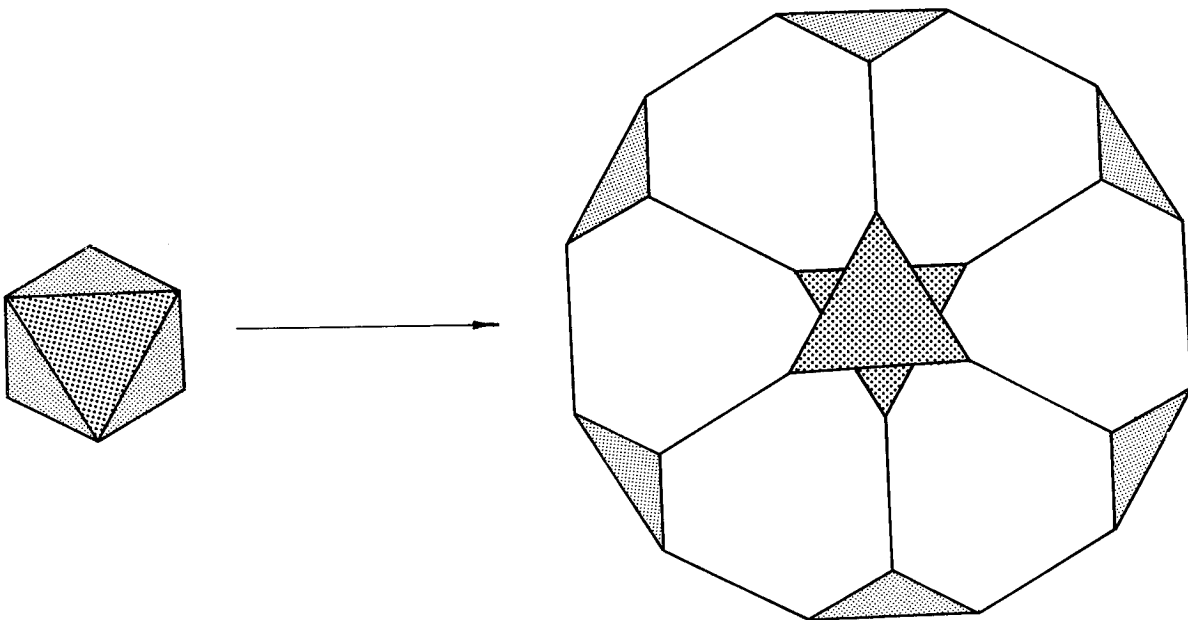


Figure 2.25

Geometric Transformations

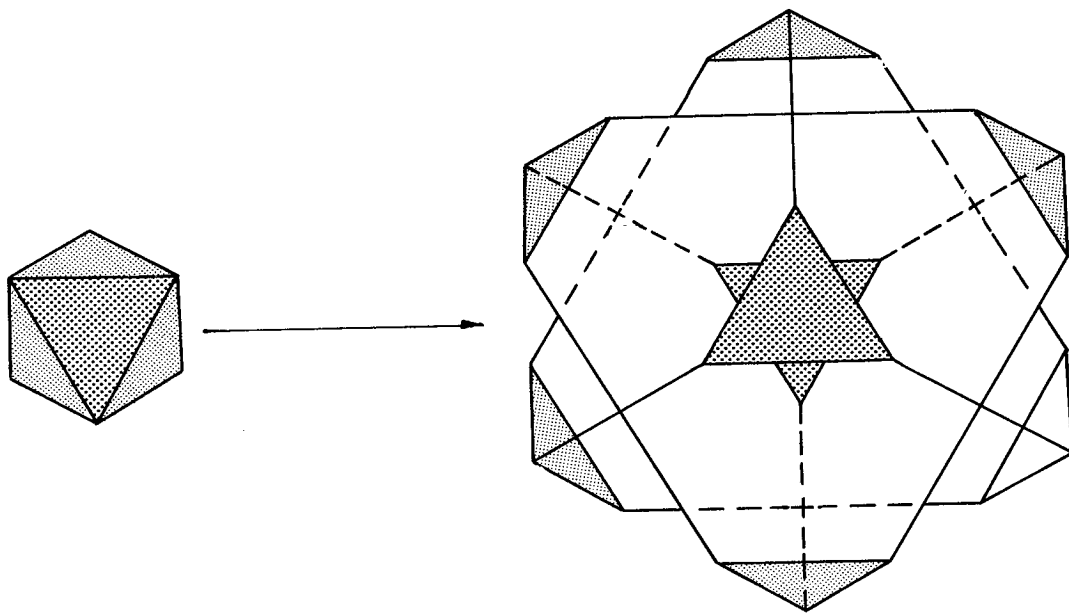


Face Rotation Translation Transformation



Element Rotation Translation Transformation

Figure 2.25 (cont.)
Geometric Transformations



Vertice Rotation-Translation Transformation

Figure 2.25 (cont.)
Geometric Transformations

2.7 Mathematical Model For Polyhedral Transformation

This portion of the investigation was restricted to an investigation of those convex polyhedra shown in Tables 2.10a and 2.10b and Figure 2.26. Figure 2.26 illustrate the various polyhedra; the numbers shown in parentheses are the coordinates of the vertex indicated by the arrow.

The transformations were limited to the expansion by center of face rotation-translation method.

It was further assumed that each face was able to move independently during the transformation. In other words, vertices of two or more faces need not be joined together during the transformation.

A unit edge was assumed for all figures.

Table 2.10a

Transformation of Polyhedra I

Platonic and Archimedean Forms

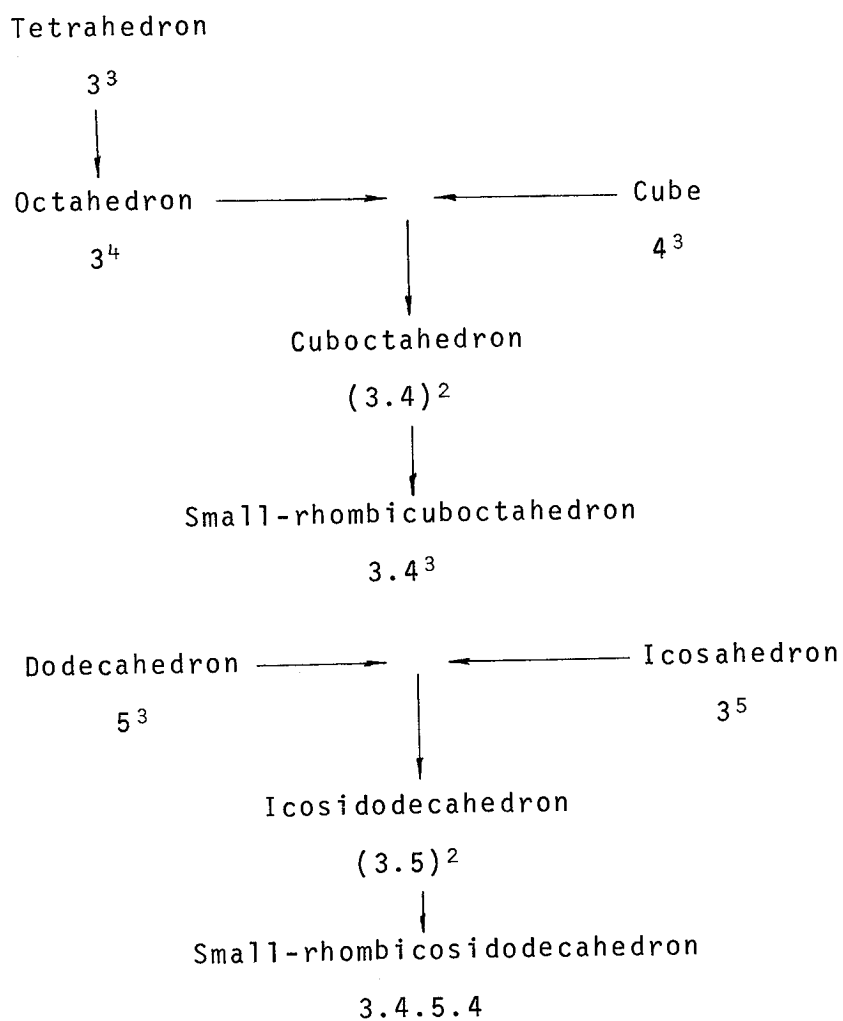


Table 2.10b

Transformation of Polyhedra II

Archimedean Dual Forms

Triakis Tetrahedron

V.3.6²



Rhombic Dodecahedron

V.(3.4)²

Tetrakis Hexahedron →

V.4.6²

← Triakis Octahedron

V.3.8²



Trapezoidal Icositetrahedron

V.3.4³

Pentakis Dodecahedron →

V.5.6²

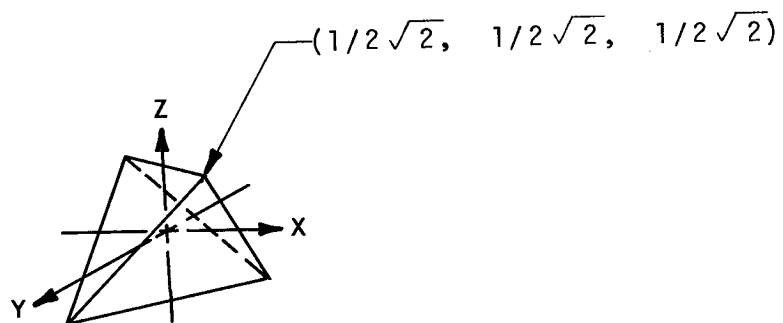
← Triakis Icosahedron

V.3.10²

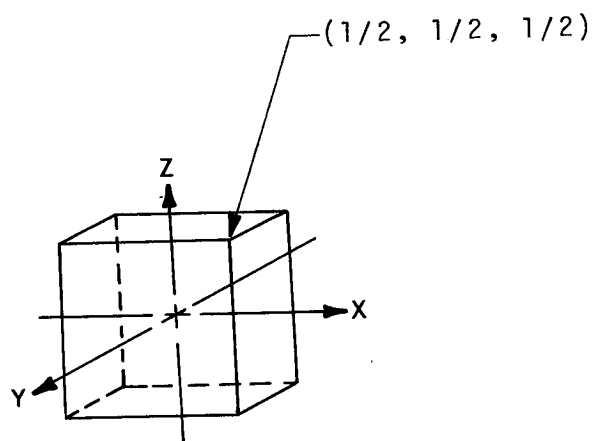


Trapezoidal Hexecontrahedron

V.3.4.5.4

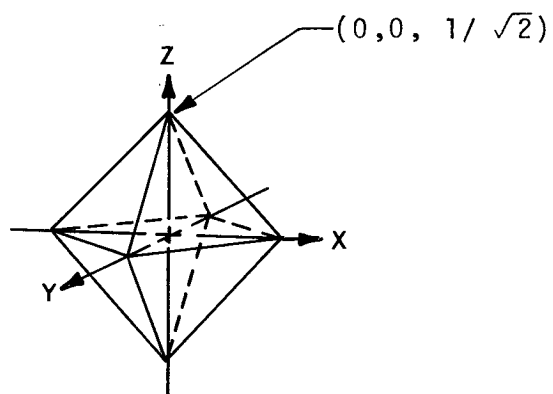


Tetrahedron

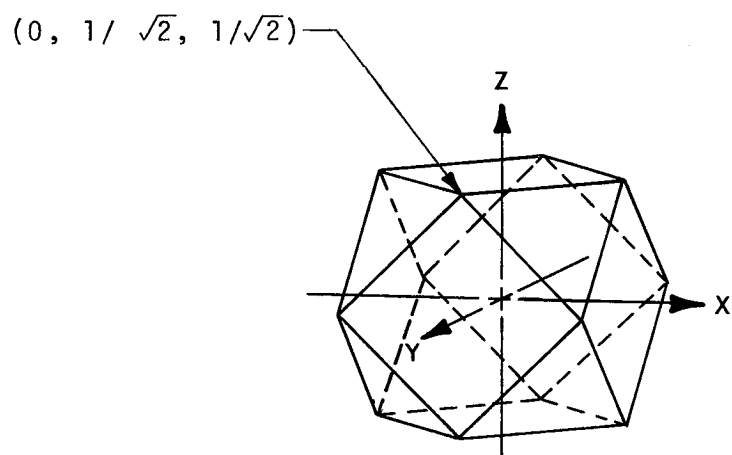


Cube

Figure 2.26

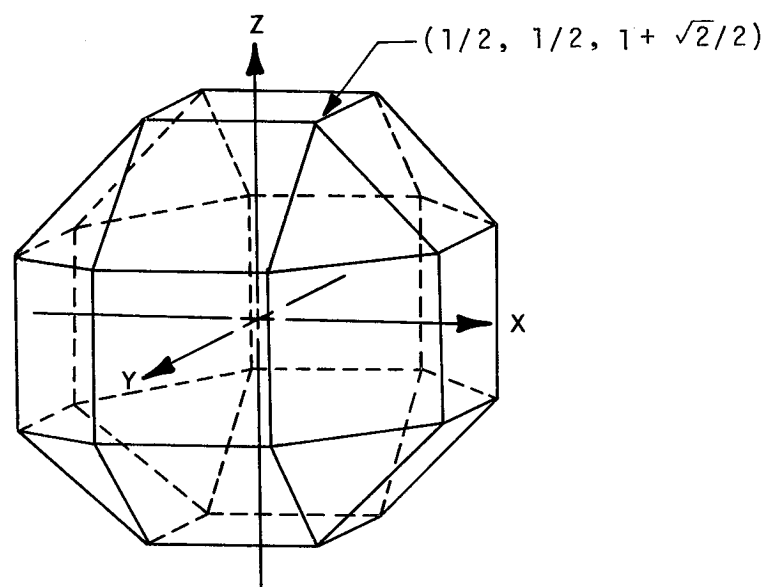


Octahedron



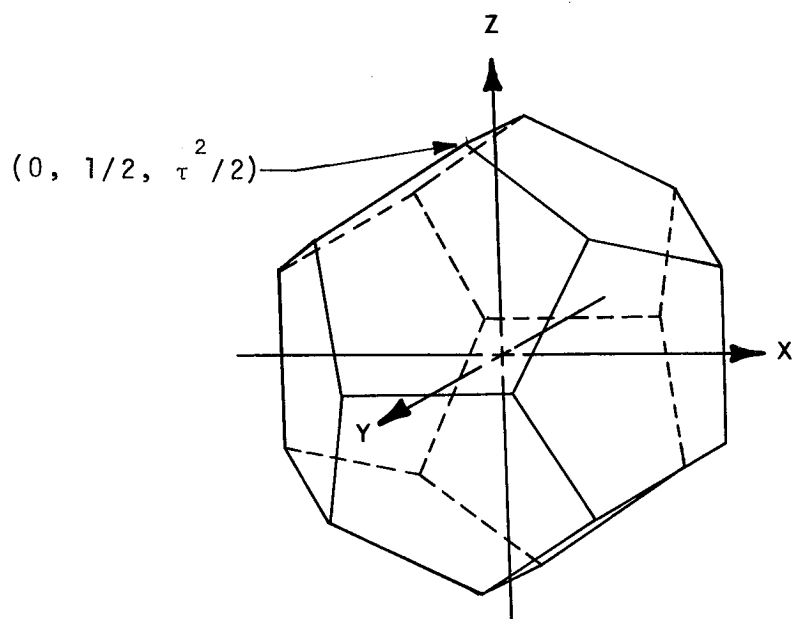
Cuboctahedron

Figure 2.26 (cont)

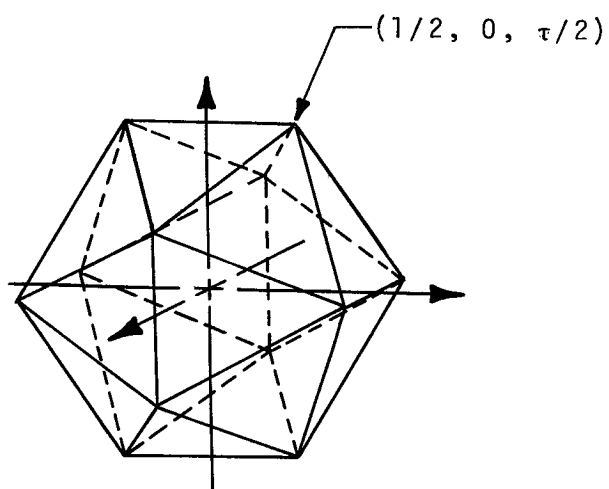


(Small) Rhombicuboctahedron

Figure 2,26 (cont)

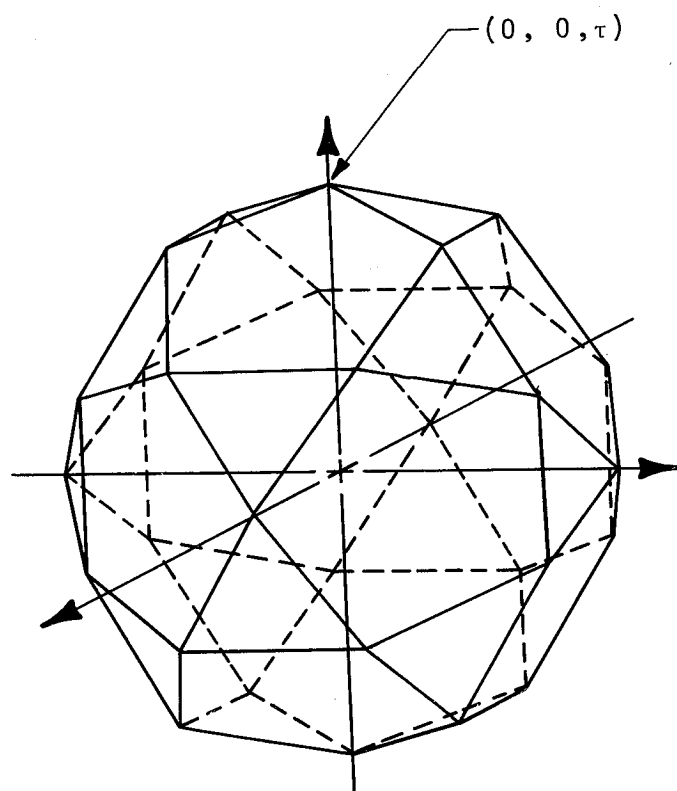


Dodecahedron



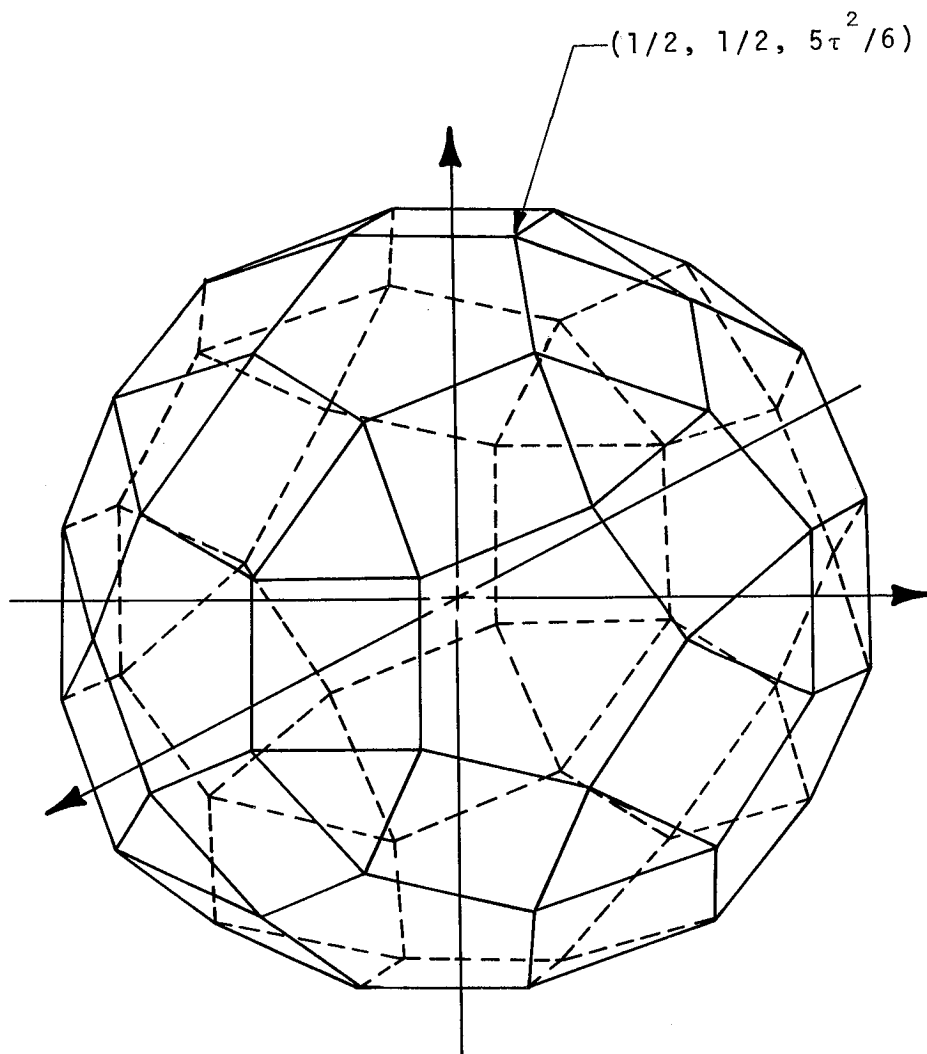
Icosahedron

Figure 2.26 (cont)



Icosidodecahedron

Figure 2.26 (cont)



(small) Rhombicosidodecahedron
Figure 2.26 (cont)

Preliminary investigation of the various polyhedra resulted in the formulation of two theories which seemed to offer possibilities for the prediction of polyhedral transformations in general.

The first theory was suggested by the nature of the transformations themselves. As has been previously stated, the transformations consist of a rotation and a translation of each polyhedral face. If the amount of rotation and the amount of translation of each of the faces could be determined for the polyhedra shown in Table 2.9, these results might suggest a pattern which could be used for the transformation of other polyhedra.

The second theory resulted from the fact that the vertices of the polyhedra seemed to move along the surface of an imaginary right circular cylinder which could be projected from each of the faces of the polyhedron. By inspection of the polyhedra which had already been transformed, it was decided that the transformation was complete when adjacent cylinders no longer intersected. The determination of the coordinates of the point at which the cylinders no longer intersect would suggest a method which could be used to predict the transformation of any of the polyhedra; the coordinates of this end point of the intersection of the cylinders would determine the vertices of the figure formed by the transformation.

The calculations were directed at determining relationships among the sides, vertices, faces, edges, and centers of the polyhedra. This resulted in the establishment of relationships which are shown in Tables 2.11 and 2.12.

Table III shows the relationship between the center angles* and the dihedral angles of the polyhedra; in all cases, the center angles are simply the supplement of the dihedral angles. Table 2.11 also shows the relationship between the transformation angles** of the original polyhedron and the dihedral angles of the new figure. The transformation angles were calculated by using the formula for determining the angle between two lines. Table 2.11 shows that the sum of the transformation angles of the figures which are paired together to form a new figure (the pairs are shown by the brackets in Table 2.11) are related to the dihedral angle of the new figure. (For the transformation of the tetrahedron, cube, and octahedron, the sum of the transformation angles equals the dihedral angle of the new figure; for the dodecahedron and the icosahedron, the sum is equal to the dihedral angle minus ninety degrees.)

Table 2.12 shows the number of faces, edges, vertices,

*Center angles are those angles formed by adjacent axis lines from the center of the figure to the center of the faces.

**Transformation angles are those angles formed by the line from the center to a vertex of the original figure and the line from the center to a corresponding vertex of the new figure.

and the number of faces meeting at each vertex; these appear to be related. In particular, the number of edges of the new figure is equal to twice the number of edges of the original figure.

According to R. Buckminster Fuller,* the number of sides of the face that is formed by the transformation will be equal to the number of faces meeting at each vertex of the original figure. This assumption was used to determine the types of faces that would be formed by the transformations; they are shown in the last column of Table 2.12. The transformations of the cuboctahedron and the icosidodecahedron were predicted by using this assumption and the one given in the paragraph above.

It was also noticed during this phase of the investigation that if the faces which form the dihedral angle of the original figure are of the same type, then the angle formed by joining the vertices of the new figure will be equal to the dihedral angle of the original figure.

It was then decided to determine whether there were any relationships among the figures in regard to the angles through which the faces rotate and the distances through which they translate during the transformations. Table 2.13 shows the transformation of the coordinates of one vertex of the original figure into the corresponding vertex of the

*Fuller, R. B., 1., p. 42.

Table 2.11
Angle Relationships

Figure	Dihedral Angle	Center Angle	Transformation Angle	Sum	Dihedral Angle of New Figure
Tetrahedron	70°32'	109°28'	54°44' *	109°28'	109°28'
Cube	90°	90°	90°	125°16'	125°16'
Octahedron	109°28'	70°32'	35°16'		
Dodecahedron	116°34'	63°26'	20°55'	52°37'	142°37'
Icosahedron	138°11'	41°49'	31°42'		

*Two tetrahedra transformed together form the octahedron.

Table 2.12
Face-Edge-Vertex Relationships

Figure	Faces	Edges	Vertices	Number of Faces at Each Vertex	New Figure Required Number of Edges and Types of Faces
Tetrahedron	4	6	4	3	12 triangular
Cube	6	12	8	3	24 triangular
Octahedron	8	12	6	4	24 square
Cuboctahedron	14	24	12	4	48 square
S. Rhombicuboctahedron	26	48	24	4	96 square
Dodecahedron	12	30	20	3	60 triangular
Icosahedron	20	30	12	5	60 pentagonal
Icosidodecahedron	32	60	30	4	120 square
S. Rhombicosidodecahedron	62	120	60	4	240 square

new figure. Because of the symmetry of the polyhedra, it was only necessary to determine the amount of rotation and the amount of translation for one face of each polyhedron, except in those cases where the polyhedra have several types of faces. For these, it was necessary to determine the amount of rotation and translation for each type of face.

By studying models which had been constructed and by using the relationship for the angle between two lines, the angles of rotation for each figure were determined. The results are shown in Table 2.14. The relationship may be stated by the following formula:

$$\begin{aligned} \text{angle of rotation} &= 180/n, \\ \text{where "n" equals the number} & \qquad \qquad \qquad 2.1 \\ \text{of sides of the face.} \end{aligned}$$

The distances through which the faces translate were then calculated for each figure, and the results are shown in Table 2.15. At this time, simple hand analysis would seem to indicate that there is no empirical relationship among the figures in regard to the amount of translation of the faces. Since it was necessary to be able to specify both the amount of rotation and the amount of translation, the rotation translation theory appeared to be of little use in solving the problem.

It was then decided to investigate the second theory. Figure 2.27a shows cylinders projected onto two adjacent faces of an octahedron. The point X' indicates the position of a vertex of the new figure.

Table 2.13
Vertex Transformations *

Tetrahedron → Octahedron

$$(1/2\sqrt{2}, 1/2\sqrt{2}, 1/2\sqrt{2}) \rightarrow (0, 0, 1/\sqrt{2}) **$$

Cube → Cuboctahedron

$$(1/2, 1/2, 1/2) \rightarrow (0, 1/\sqrt{2}, 1/\sqrt{2})$$

Octahedron → Cuboctahedron

$$(0, 0, 1/\sqrt{2}) \rightarrow (0, 1/\sqrt{2}, 1/\sqrt{2})$$

Cuboctahedron → Small rhombicuboctahedron

$$(0, 1/\sqrt{2}, 1/\sqrt{2}) \rightarrow (1/2, 1/2, 1+\sqrt{2}/2)$$

Dodecahedron → Icosidodecahedron

$$(0, 1/2, \tau^2/2) \rightarrow (0, 0, \tau)^2$$

Icosahedron → Icosidodecahedron

$$(1/2, 0, \tau/2) \rightarrow (0, 0, \tau)$$

Icosidodecahedron → Small rhombicosidodecahedron

$$(0, 0, \tau) \quad (1/2, 1/2, 5\tau^2/6)$$

*These figures are the coordinates of one of the vertices of the original figure and the corresponding vertex of the new figure.

$$**\tau = (1+\sqrt{5})/2; \quad \tau^2 = \tau + 1$$

Table 2.14
Amount of Rotation of the
Faces of the Polyhedra

<u>Figure</u>		<u>Angle of Rotation</u>
Tetrahedron		60°
Cube		45°
Octahedron		60°
Cuboctahedron	(triangular face)	60°
	(square face)	45°
Dodecahedron		36°
Icosahedron		60°
Icosidodecahedron	(triangular face)	60°
	(pentagonal face)	36°

Table 2.15

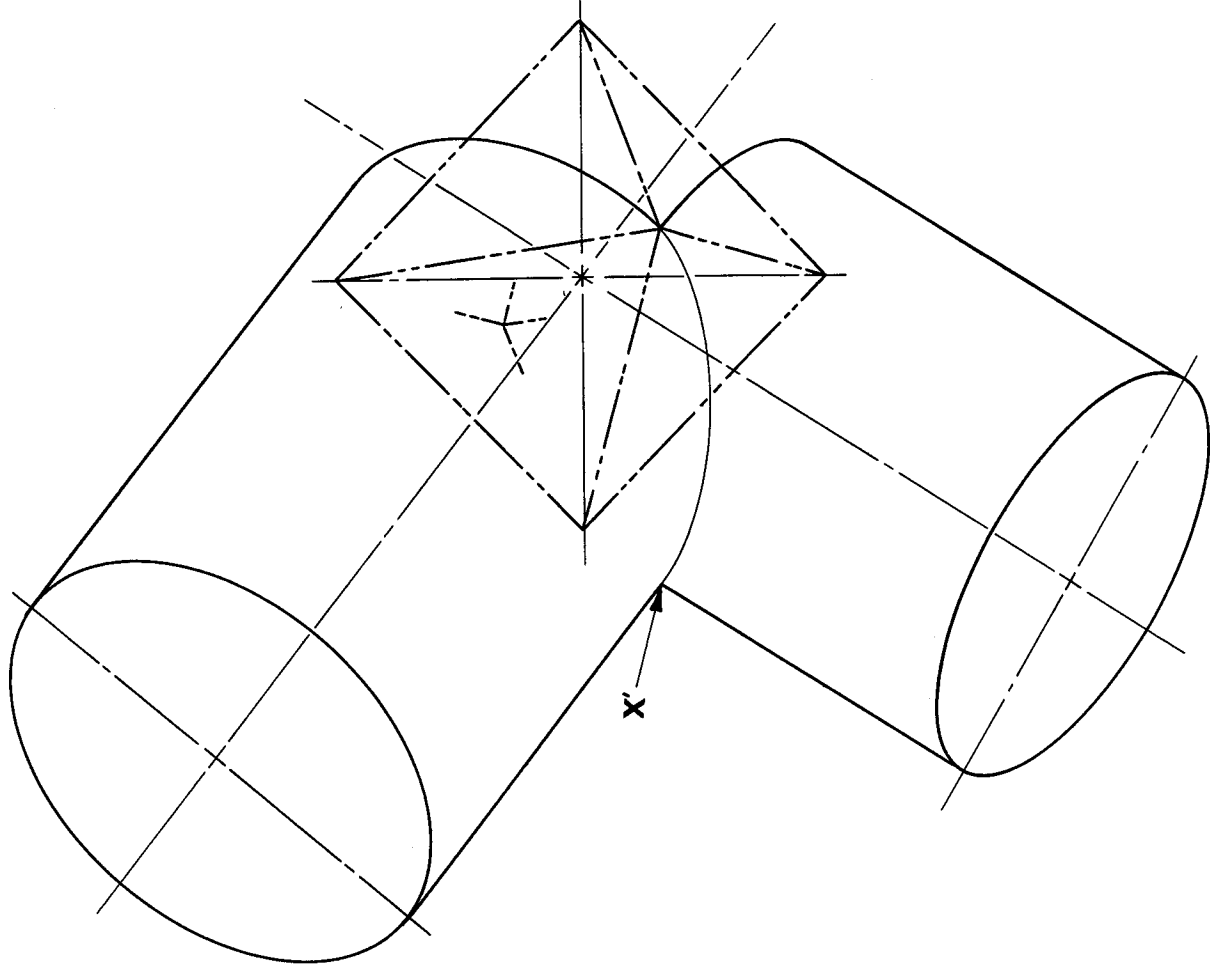
Amount of Translation of the
Faces of the Polyhedra

Figure	Distance from Center to Center of Face		Translation Distance Faced Moved From Center
	Before	After	
Tetrahedron	$1/2\sqrt{6}$	$1/\sqrt{6}$	$1/2\sqrt{6}$
Cube	$1/2$	$1/\sqrt{2}$	$1/\sqrt{2} - 1/2$
Octahedron	$1/\sqrt{6}$	$2/3$	$1/\sqrt{6}$
Cuboctahedron *	$1/\sqrt{2}$	$1/\sqrt{2} + 1/2$	$1/2$
	$2/3$	$(11+6\sqrt{2})/12$	$\frac{(11+6\sqrt{2})}{12} - \sqrt{\frac{2}{3}}$
Dodecahedron	$\tau^{5/2} / 2(5^{1/4})$	$\frac{\tau^2 - (1.76)^2}{2}$	α^2
Icosahedron	$\tau^2 / 2\sqrt{3}$	$\tau^2 / \sqrt{3}$	$\tau^2 / 2\sqrt{3}$
Icosidodecahedron *	$\tau^2 - (1.76)^2$	$\frac{\sqrt{25\tau^4 - 9(1.76)}}{6}$	β^3
	$\tau^2 / \sqrt{3}$	$\frac{\sqrt{25\tau^4 - 6}}{6}$	$\frac{\sqrt{25\tau^4 - 6}}{6} - \frac{\tau^2}{\sqrt{3}}$

*The second line of figures refer to the triangular faces.

$$**\alpha = \left(\frac{\tau^2 - (1.76)^2}{2} \right) - \frac{\tau^{5/2}}{2(5^{1/4})}$$

$$***\beta = \frac{\sqrt{25\tau^4 - 9(1.76)}}{6} - \left(\frac{\tau^2 - (1.76)^2}{2} \right)$$



Octahedron with Intersecting Cylinders

Figure 2.27a

Two cylinders intersect in the manner shown in Figure 2.27b. The arc XX' represents the curve of intersection; point X represents a vertex of the original figure and point X' represents the point along the curve which is a maximum distance from the center of the polyhedron. This point is one of the vertices of the figure formed by the transformation. In order to determine this point it was necessary to specify the equations of the adjacent cylinders as shown in Figure 2.27b and the equation of a plane.

Each of the vertices of the figure formed by the transformation lies in a plane which passes through the center of the original polyhedron and the axes of its adjacent faces. Each of these vertices can be determined by finding the point of intersection of the plane and the cylinders. The intersection of the plane and the cylinders. The intersection can be found by solving the set of simultaneous equations consisting of the two equations of the adjacent cylinders and the equation of the plane.

The general equation used for the cylinders was the following: $r^2 = ax^2 + by^2 + cz^2 + dxy + exz + fyz$, where "r" is the radius of the cylinder and "a", "b", and "c" are constants which have the following values:

$$\begin{aligned} a &= \frac{m^2 + n^2}{l^2 + m^2 + n^2} & b &= \frac{l^2 + n^2}{l^2 + m^2 + n^2} & c &= \frac{l^2 + m^2}{l^2 + m^2 + n^2} \\ d &= \frac{-2lm}{l^2 + m^2 + n^2} & e &= \frac{-2ln}{l^2 + m^2 + n^2} & f &= \frac{-2mn}{l^2 + m^2 + n^2} \end{aligned} \quad 2.2$$

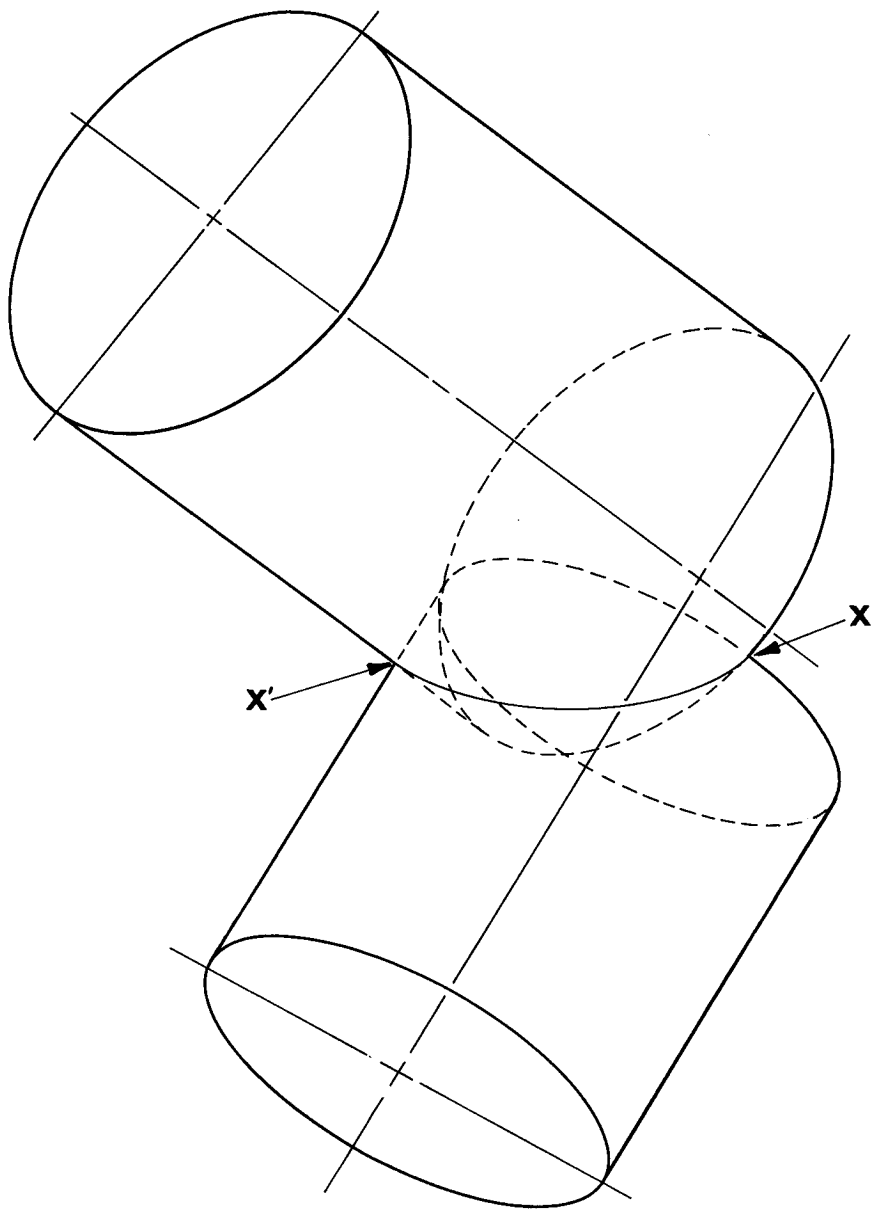


Figure 2.27b

and where "l", "m", and "n" are the direction numbers of the line which joins the center of the polyhedron and the center of the face of the polyhedron.

The general equation used for the planes was derived from the general form of the equation of a plane passing through the origin: $ax + by + cz = 0$, where "a", "b", and "c" are direction numbers of the normal to the plane. 2.3

In order to use the above equations, it is necessary to specify the coordinates of the vertices of the original figure and the number of edges of the original figure. Table 2.16 shows that the number of vertices of the new figure can be determined by specifying the number of edges of the original figure; they are equal. This number indicates the number of sets of simultaneous equations which must be solved in order to specify all of the vertices of the new figure.

If the vertices of a polyhedron are known, the equations of the adjacent cylinders and the equation of the plane can be determined by using the above general equations. The solution of this set of equations yields two vertices of the new figure. This process must be repeated until all combinations of pairs of adjacent cylinders are used.

Shown below is the complete transformation for the tetrahedron:

Vertices of the tetrahedron:

$(1/2\sqrt{2}, 1/2\sqrt{2}, 1/\sqrt{2})$	$(1/2\sqrt{2}, -1/2\sqrt{2}, -1/2\sqrt{2})$
$(-1/2\sqrt{2}, 1/2\sqrt{2}, -1/2\sqrt{2})$	$(-1/2\sqrt{2}, -1/2\sqrt{2}, -1/2\sqrt{2})$

Table 2.16

Relationship among the Edges of the Original
Figure and the Vertices of the New Figure

Original Figure	Number of Edges	New Figure	Number of Vertices
Tetrahedron	6	Octahedron	6
Cube	12	Cuboctahedron	12
Octahedron	12	Cuboctahedron	12
Cuboctahedron	24	Small-rhombicuboctahedron	24
Dodecahedron	30	Icosidodecahedron	30
Icosahedron	30	Icosidodecahedron	30
Icosidodecahedron	60	Small-rhombicosidodecahedron	60

Equations of the projecting cylinders (one for each face):

- | | | |
|-----|--|-----|
| (1) | $x^2 + y^2 + z^2 + xy + xz - yz = 1/2$ | 2.4 |
| (2) | $x^2 + y^2 + z^2 + xy - xz + yz = 1/2$ | 2.5 |
| (3) | $x^2 + y^2 + z^2 - xy + xz + yz = 1/2$ | 2.6 |
| (4) | $x^2 + y^2 + z^2 - xy - xz - yz = 1/2$ | 2.7 |

Equations of the planes for the cylinder pairs shown

in parentheses:

- | | | |
|-------|----------|------|
| (1,2) | $x = -y$ | 2.8 |
| (1,3) | $x = -z$ | 2.9 |
| (1,4) | $y = z$ | 2.10 |
| (2,3) | $y = -z$ | 2.11 |
| (2,4) | $x = z$ | 2.12 |
| (3,4) | $x = y$ | 2.13 |

These equations yield the following six sets of simultaneous equations:

- | | | |
|-----|--|--------------------------------|
| 1.) | $x^2 + y^2 + z^2 + xy + xz - yz = 1/2$ | 2.4 |
| | $x^2 + y^2 + z^2 + xy - xz + yz = 1/2$ | 2.5 |
| | $x = -y$ | 2.8 |
| | $(0, 0, 1/\sqrt{2})$ | $(0, 0, -1/\sqrt{2})$ |
| | $(1/\sqrt{2}, -1/\sqrt{2}, 0)$ | $(-1/\sqrt{2}, 1/\sqrt{2}, 0)$ |
| 2.) | $x^2 + y^2 + z^2 + xy + xz - yz = 1/2$ | 2.4 |
| | $x^2 + y^2 + z^2 - xy + xz + yz = 1/2$ | 2.6 |
| | $x = -z$ | 2.9 |
| | $(0, 1/\sqrt{2}, 0)$ | $(0, -1/\sqrt{2}, 0)$ |
| | $(-1/\sqrt{2}, 0, 1/\sqrt{2})$ | $(1/\sqrt{2}, 0, -1/\sqrt{2})$ |

3.)	$x^2 + y^2 + z^2 + xy + xz - xy = 1/2$	2.4
	$x^2 + y^2 + z^2 = xy - xz - yz = 1/2$	2.7
	$y = z$	2.10
	$(1/\sqrt{2}, 0, 0)$	$(-1/\sqrt{2}, 0, 0)$
	<u>$(0, 1/\sqrt{2}, 1/\sqrt{2})$</u>	<u>$(0, -1/\sqrt{2}, -1/\sqrt{2})$</u>
4.)	$x^2 + y^2 + z^2 - xy + xz + yz = 1/2$	2.6
	$x^2 + y^2 + z^2 - xy - xz - yz = 1/2$	2.7
	$x = y$	2.13
	$(0, 0, 1/\sqrt{2})$	$(0, 0, -1/\sqrt{2})$
	<u>$(1/\sqrt{2}, 1/\sqrt{2}, 0)$</u>	<u>$(-1/\sqrt{2}, -1/\sqrt{2}, 0)$</u>
5.)	$x^2 + y^2 + z^2 + xy - xz + yz = 1/2$	2.4
	$x^2 + y^2 + z^2 - xy - xz - yz = 1/2$	2.7
	$x = z$	2.12
	$(0, 1/\sqrt{2}, 0)$	$(0, -1/\sqrt{2}, 0)$
	<u>$(1/\sqrt{2}, 0, 1/\sqrt{2})$</u>	<u>$(-1/\sqrt{2}, 0, -1/\sqrt{2})$</u>
6.)	$x^2 + y^2 + z^2 + xy - xz + yz = 1/2$	2.5
	$x^2 + y^2 + z^2 - xy + xz + yz = 1/2$	2.6
	$y = -z$	2.11
	$(1/\sqrt{2}, 0, 0)$	$(-1/\sqrt{2}, 0, 0)$
	<u>$(0, 1/\sqrt{2}, -1/\sqrt{2})$</u>	<u>$(0, -1/\sqrt{2}, 1/\sqrt{2})$</u>

The solutions of these sets are shown in the parentheses; it should be observed that each set yields four solutions (two of which are the coordinates of the vertices of the new figure and two which are not). It should also be noticed that the solutions for the last three sets of equations are duplicates of the first three. If the equations are selected with regard to the symmetry of the figure, it will be necessary to use only half the number of sets of equations

as is indicated in Table 2.16. Because of the symmetry of the figures, the solutions will always be duplicated.

At this point, a decision must be made as to which solutions are valid ones for the transformations. In the case of the regular polyhedra, this can be done by specifying another condition which the points must satisfy. For the regular polyhedra, each vertex of the new figure will lie in a plane which passes through the center of the figure and the edge between the adjacent faces under consideration. If the equation of this plane is added to the set of equations, two of the solutions (for the tetrahedron--those above which are underlined) will be eliminated and the other two will be vertices of the new figure. For example, the equation of this plane for the cylinder pair (1,2) above is $x = y$. Adding this equation to the set of equations (number 1) for the tetrahedron eliminates the underlined solutions.

For the semi-regular polyhedra, such as the cuboctahedron, the selection of the proper solutions for specifying the vertices of the new figure must be made on the basis of the physical restrictions of the figure.

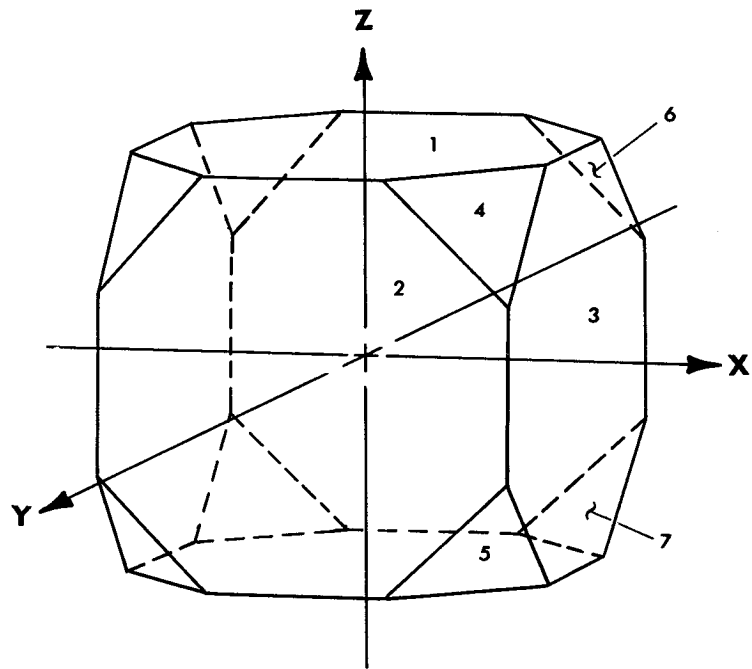
In all cases, the solution of each set of equations will yield four points of intersection (which will be in pairs because of the symmetry of the polyhedra); it will be necessary to analyze these groups of pairs in order to determine which fits the physical restrictions of the figures. For example, the transformation of the cuboctahedron yields 96

solutions, half of which are identical to the other half; they are of two forms: $(1/2, 1/2, 1+\sqrt{2}/2)$ and $(1/2, 1/2, 1-\sqrt{2}/2)$, where the order and signs of the numbers can be interchanged. Simple inspection of these solutions reveals that those of the second type could not specify the vertices of the new figure. (The figure would be smaller than the cuboctahedron).

It should be pointed out that this method is suitable only for those convex polyhedra whose faces are equilateral and equiangular. If the faces are not of this type, the cylinders would assume a different form; ie., they would not be right circular cylinders.

Since the purpose of this research was to find a theoretical concept for the transformation of polyhedra,* it was deemed advisable to try the method on a figure which had not been previously transformed. This was done for the truncated cube shown in Figure 2.28. Shown below are the calculations for this transformation. Because of the symmetry of the truncated cube, it is necessary to specify only seven equations for the projecting cylinders (three for the octagonal faces and four for the triangular faces). Inspection of Figure 2.28 shows that if a plane of symmetry was drawn through the polyhedron (parallel to and passing through the z-axis), the polyhedron reflects itself about the plane. Because of this

*This was done by transforming those in Table 2.9.



Truncated Cube

Figure 2.28

the equations of the cylinders for those faces to the right of the plane are identical to those to the left. The same situation holds for the equations of the planes.

Equations of the cylinders:

- (1) $x^2 + y^2 = 1.69$
- (2) $x^2 + z^2 = 1.69$
- (3) $y^2 + z^2 = 1.69$
- (4) $x^2 + y^2 + z^2 - xy - xz - yz = 1/2$
- (5) $x^2 + y^2 + z^2 - xy + xz + yz = 1/2$
- (6) $x^2 + y^2 + z^2 + xy - xz + yz = 1/2$
- (7) $x^2 + y^2 + z^2 + xy + xz - yz = 1/2$

Equations of the planes:

- (1,2) $x = 0$
- (1,3) $y = 0$
- (2,3) $z = 0$
- (1,4)
(1,5) $x = y$
- (2,4)
(2,6) $x = z$
- (3,4)
(3,7) $y = z$
- (1,6)
(1,7) $x = -y$
- (2,5)
(2,7) $x = -z$
- (3,5)
(3,6) $y = -z$

The numbers in parentheses which precede the equations refer to the faces as they are numbered in Figure 2.28.

Application of these equations results in fifteen sets* of simultaneous equations which must be solved in order to specify the complete transformation. Two examples of these sets and their solutions are given below.

$$1.) \quad x^2 + y^2 = 1.69$$

$$x^2 + z^2 = 1.69$$

$$x = 0$$

$$(0, 1.3, 1.3) \qquad (0, -1.3, -1.3)$$

$$(0, 1.3, -1.3) \qquad (0, -1.3, 1.3)$$

$$2.) \quad x^2 + y^2 = 1.69$$

$$x^2 + y^2 + z^2 - xy - xz - yz = 1/2$$

$$x = y$$

$$(.92, .92, 1.6) \qquad (-.92, -.92, 1.6)$$

The figures in the parentheses are the coordinates of the points which represent vertices of the new figure. For set number one, all four of the solutions represent vertices of the new figure.* For set number two, the two points shown are the ones which represent the vertices. Because the truncated cube has 36 edges, the complete solution of all sets of equations results in 36 points which represent vertices of the new figure. Twelve of the points

*The truncated cube has 36 edges, but because of the symmetry of the figure only 18 sets of equations are needed. Since each octagonal face is perpendicular to one of the coordinate axes, all four of these solutions are valid vertices. For this reason, only 15 sets of equations are needed.

are of the form: $(1.3, 1.3, 0)$; the other twenty-four are of the form: $(.92, .92, 1.6)$. The differences in the points result from changing the order and the signs of the coordinates. Figure 2.29 illustrates the polyhedron that is formed by the transformation of the truncated cube. It should be pointed out that analysis of the vertices of the figure that is formed indicates that the surface of the octagonal faces must undergo a distortion.

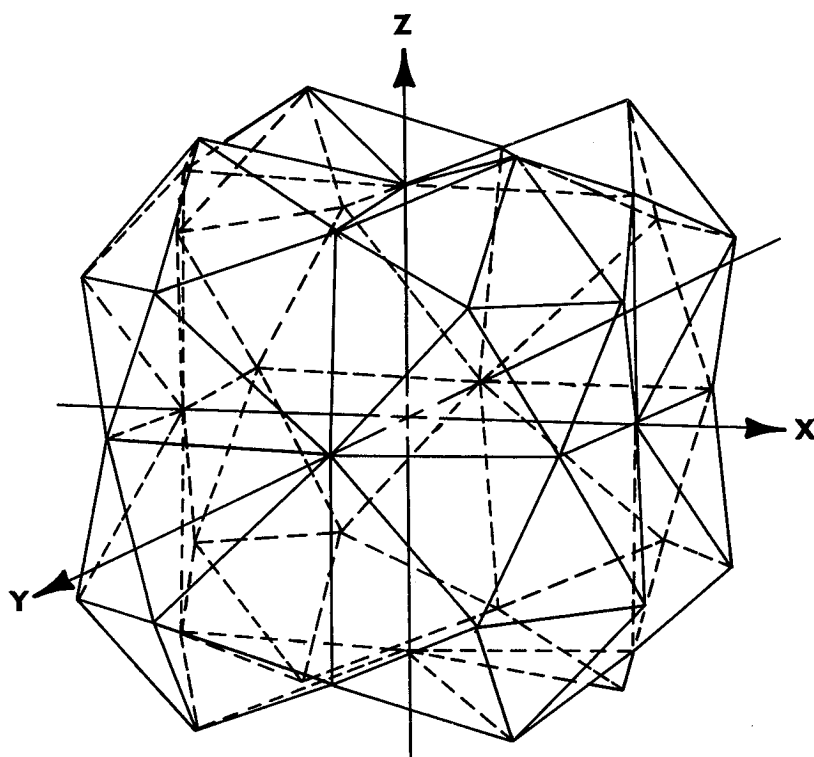


Figure 2.29

2.8 Transformation of Polyhedra

Each of the three types of polyhedral transformations may pass through a rotation of 360° in which the polyhedral form cycles from a closed state, open state, and an infold state. The path which one vertex follows is an ellipse and may be defined as follows. Figure 2.30

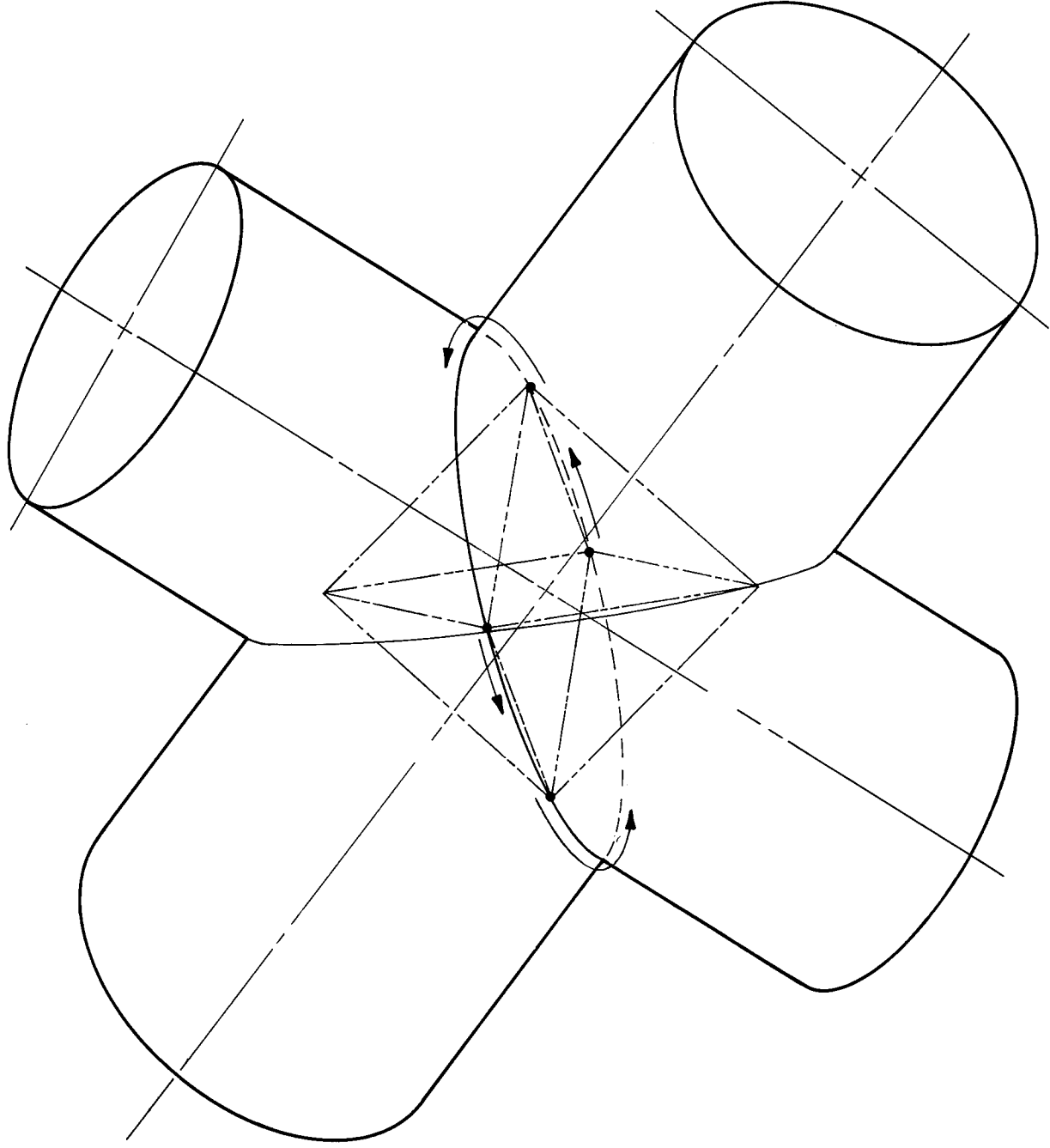


Figure 2.30

The ellipse takes the form:

$$b^2 x^2 + c^2 y^2 = c^2 b^2$$

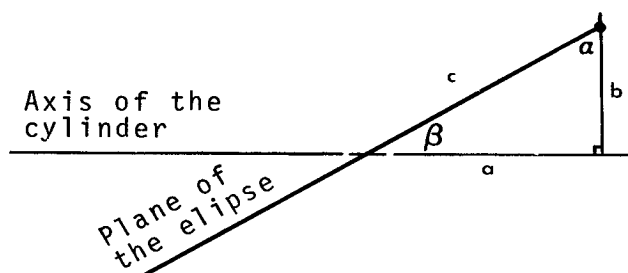
2.12

where: $b = 1/2$ minor axis of the ellipse
(the radius of the cylinder)

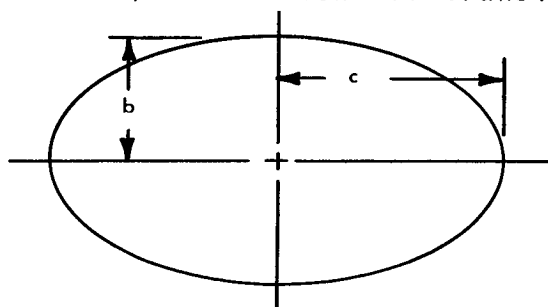
$c = 1/2$ major axis of the ellipse

and $c = \frac{b}{2 \cos \alpha}$

where: $\alpha = 1/2$ dihedral \angle of the original
polyhedron



$a = 1/2$ the extent of translation



The ellipse angle may be described as:

$$\beta = 180^\circ - (90^\circ + \alpha)$$

$$\beta = 90^\circ - \alpha$$

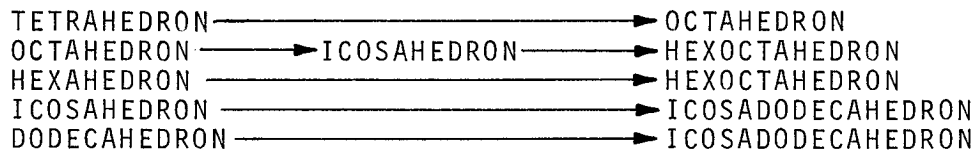
or $\sin \beta = \frac{b}{c}$

2.13

All of the regular and semi-regular polyhedral forms and many higher forms have been studied. The following illustration show the transformation of the five regular forms using two methods of expansion. The face transformation and the edge transformation. Tables 2.17 and 2.18, Figures 2.31 and 2.32.

Table 2.17

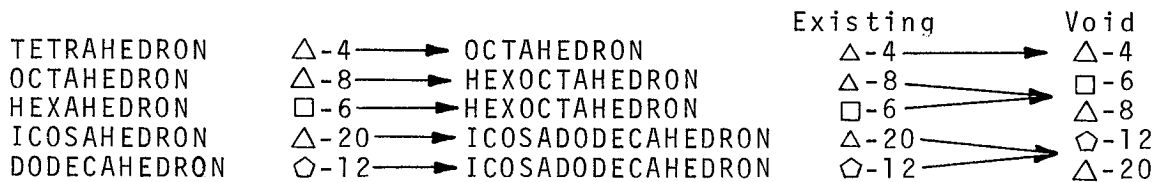
Face Transformations of
Platonic Polyhedra

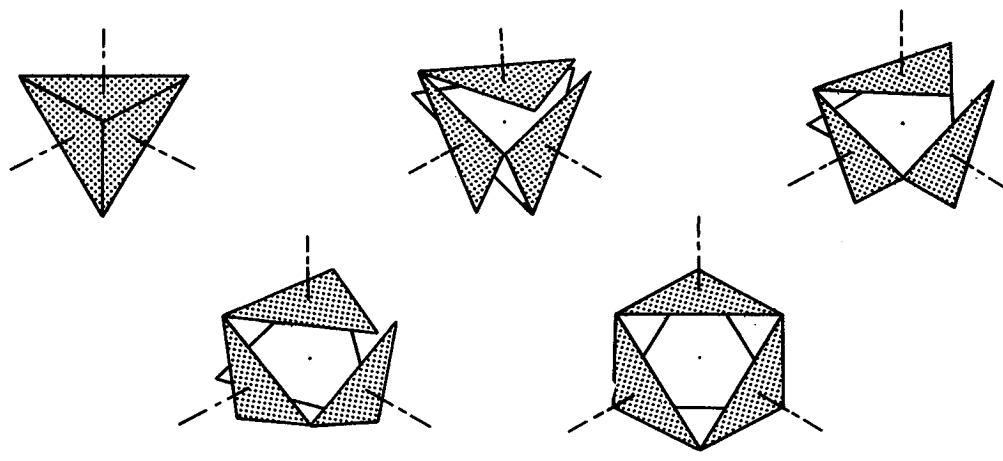


	VERTICES	FACES	EDGES	△	□	◇
TETRAHEDRON	4	4	6	4	-	-
OCTAHEDRON	6	8	12	8	-	-
HEXAHEDRON	8	6	12	-	6	-
ICOSAHEDRON	12	20	30	20	-	-
DODECAHEDRON	20	12	30	-	-	12
HEXAHEDRON	12	14	24	8	6	-
ICOSADODECAHEDRON	30	32	60	20	-	12

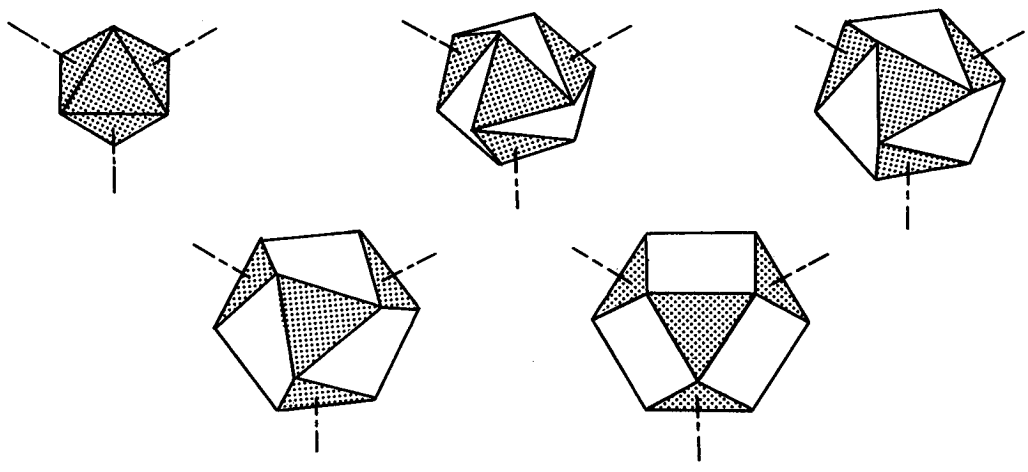
FACES
EXISTING

FACES



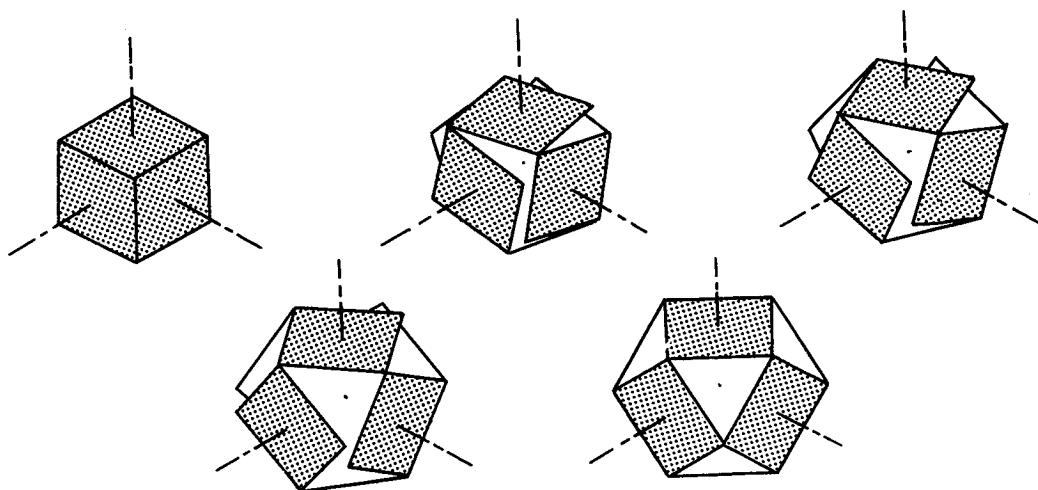


Tetrahedron → Octahedron

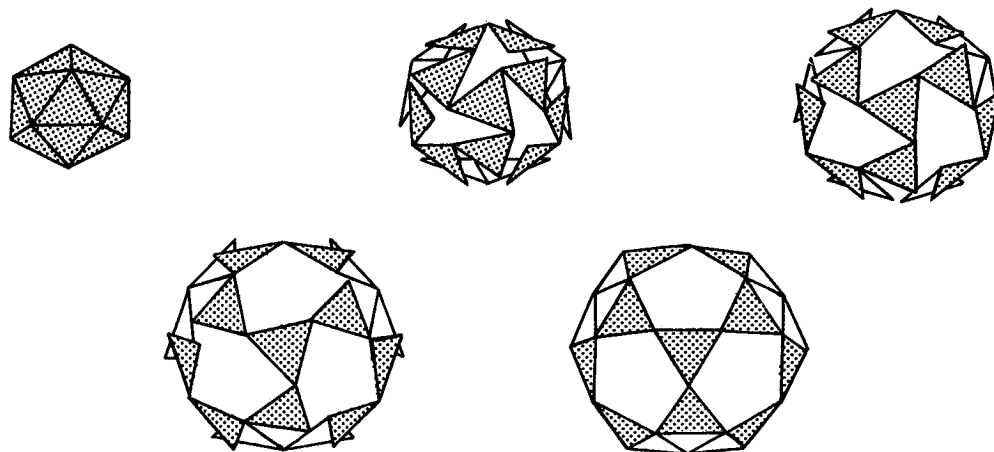


Octahedron → Hexoctahedron

Figure 2.31

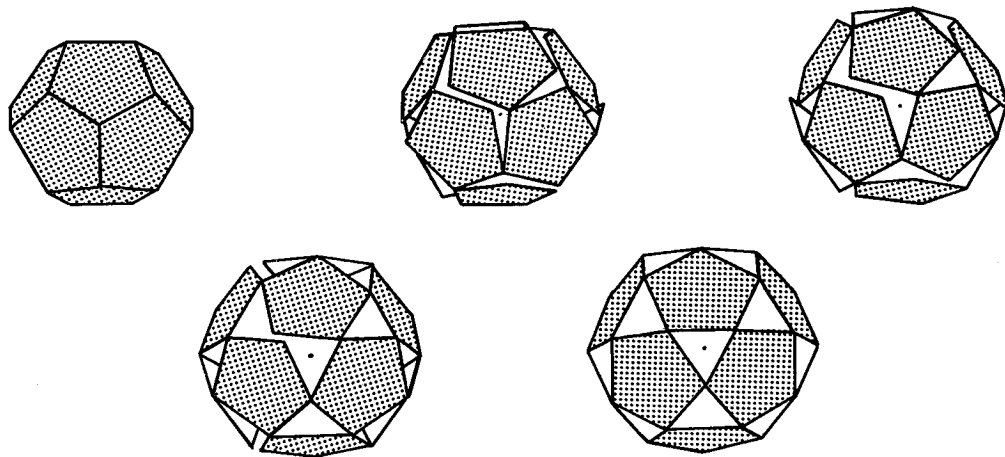


Hexahedron —————> Hexoctahedron



Icosahedron —————> Icosadodecahedron

Figure 2.31 (continued)



Dodecahedron → Icosadodecahedron

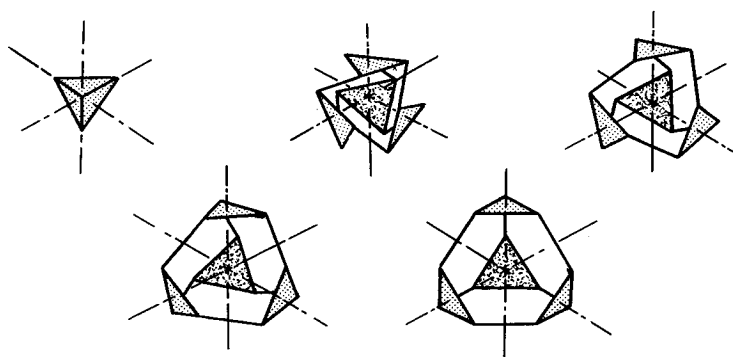
Figure 2.31 (continued)

Table 2.18

Edge Transformation of Regular Polyhedra

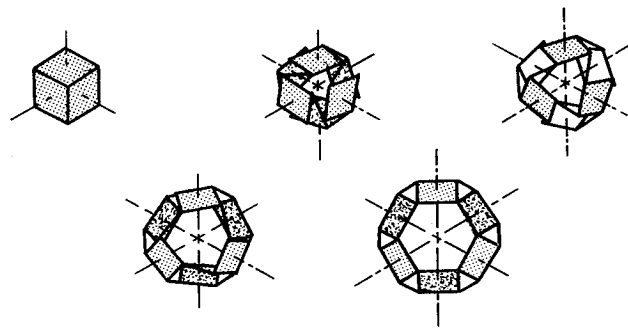
Tetrahedron → Icosahedron → Truncated Tetrahedron
 Cube → Snub Cube → Truncated Octahedron
 Octahedron → Snub Cube → Truncated Cube
 Icosahedron → Snub Dodecahedron → Truncated Dodecahedron
 Dodecahedron → Snub Dodecahedron → Truncated Icosahedron

	Vertices	Faces	Edges	3	4	5	6	8	10
Tetrahedron	4	4	6	4	-	-	-	-	-
Octahedron	6	8	12	8	-	-	-	-	-
Hexahedron	8	6	12	-	6	-	-	-	-
Icosahedron	12	20	30	20	-	-	-	-	-
Dodecahedron	20	12	30	-	-	12	-	-	-
Snub Cube	20	38	60	32	6	-	-	-	-
Snub Dodecahedron	60	92	150	80	-	12	-	-	-
Truncated Tetrahedron	12	8	18	4	-	-	4	-	-
Truncated Octahedron	24	14	36	-	6	-	8	-	-
Truncated Cube	24	14	36	8	-	-	-	6	-
Truncated Dodecahedron	60	32	90	20	-	-	-	-	12
Truncated Icosahedron	60	32	90	-	-	12	20	20	-

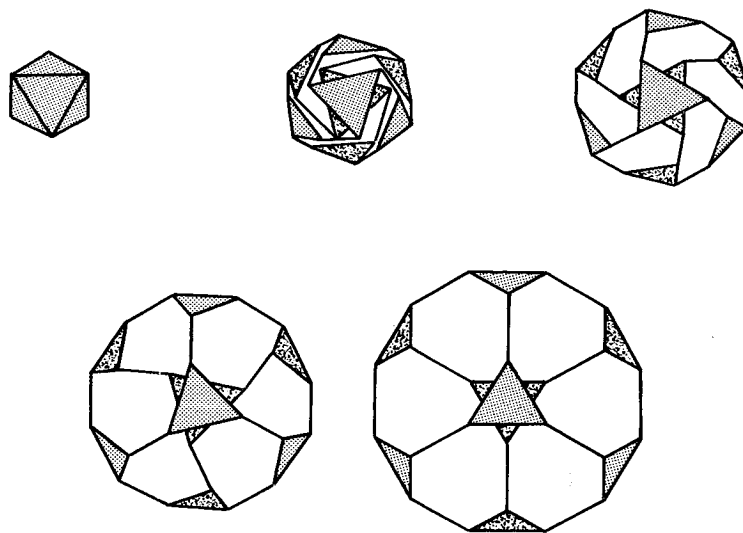


Tetrahedron → Truncated Tetrahedron

Figure 2.32

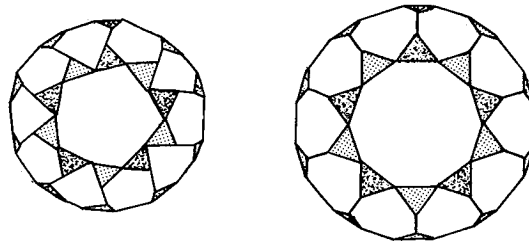


Cube → Truncated Octahedron

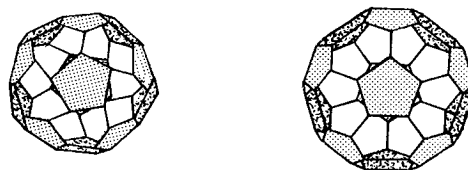
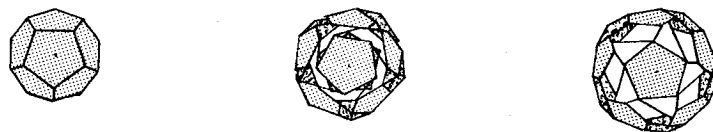


Octahedron → Truncated Cube

Figure 2.32 (cont)



Icosahedron → Truncated Dodecahedron



Dodecahedron → Truncated Icosahedron

Figure 2.32 (cont)

The investigation of the Geomertic Transformation concept for expanding tessellation and polyhedral forms is still in its infancy. Many possibilities exist in the realm of investigation and many answers must be sought. There are several methods of transforming polyhedra in closest packing which has yet to be completely documented. The concept of regeneration needs to be thoroughly investigated. Many possibilities under present investigation are the warping of a tessellation transformation into modular space frame configurations of several shapes.

It should be pointed out that in the concept of polyhedral transformation by vertice rotation-translation, the figure in its expanded state does not yield regular forms.

This tends to indicate new areas for investigations, perhaps in the realm of Star Polytopes. The in-fold state of both tessellations and polyhedra suggest study of the relationship of the Star Polygons and Star Polytopes.

Finally, the Geometric Transformation concept tends to indicate a comprehensively ordered system which may lead to a better understanding of the many areas related to the study of tessellation and polyhedral forms.

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2.10 Other Geometric Transformation Concepts

The geomertic transformation concept has shown significance as an expandable structural configuration and has lent to research in polyhedral cycling. It was noted in filling the void areas created during the transformation that if an appropriate number of faces are used instead of a single face to fill the void, a cycling effect took place. The faces to be used need to have the same length of element as the existing surfaces. They also need to be equal in number to the existing exposed sides. (i.e. the octahedron transforms into the hexoctahedron leaving square void areas. Instead of filling the voids with square surfaces, they are filled with four triangles that match the existing surfaces. The result is the original transformation axis is retained and the cycling can be repeated) Figure 2.34. It is conceivable that through locking and retransforming, the cycling could take place as many times as physically or mechanically possible.

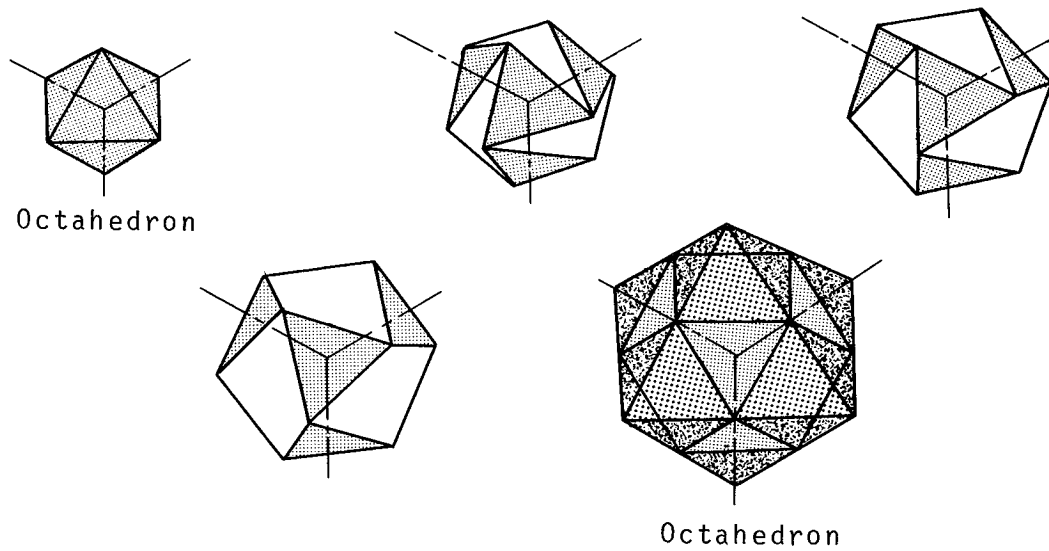
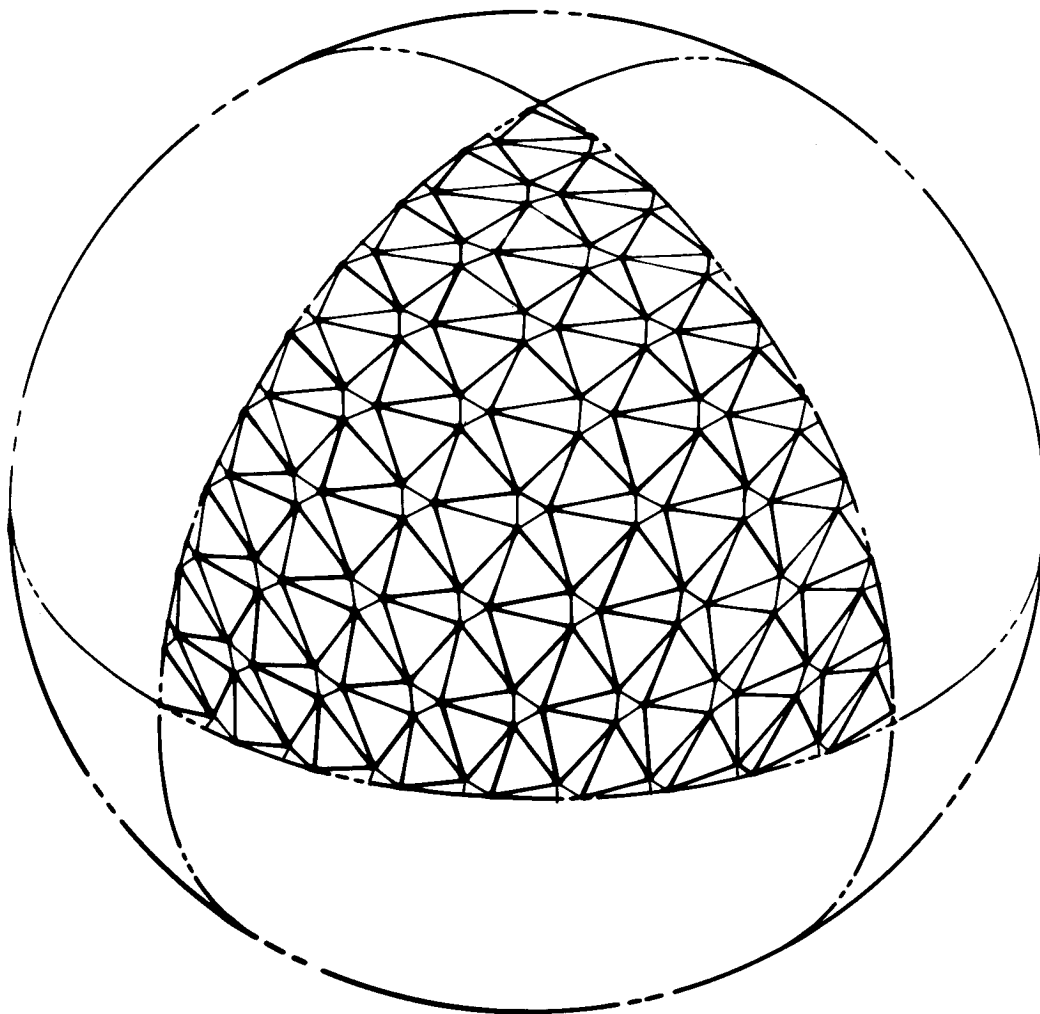
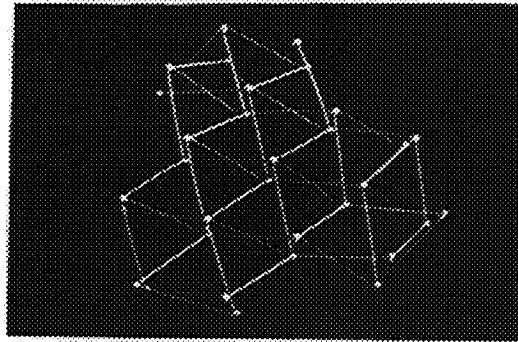


Figure 2.34
Transformation Cycling

Another such area of investigation includes the use of the tessellation net transformations, where after transformation the tessellation is warped or moved out of place into a curved or warped plan allowing for surface covering of space forms. Figure 2.35



By applying the tension and tensegrity concepts, the transformation concept has lead to investigations of space frame structures for possible space applications. Figure 2.36 illustrates a configuration where the elements of a space frame (closed packed hexagons) undergo the geometric transformation and expand from a bundle composed of the elements into a space frame configuration. The uniqueness of this structure is that it may take on any desirable shape.



Transformation Using Tension

Figure 2.36

A transformation may be done to the polyhedron in a close packed configuration where the unit cell does not transform itself. The unit cell undergoes a transformation with its adjacent cell by one of the three transformations (unit cell transformation, unit edge transformation, or unit vertex transformation.) Figure 2.37 illustrates a unit cell transformation.

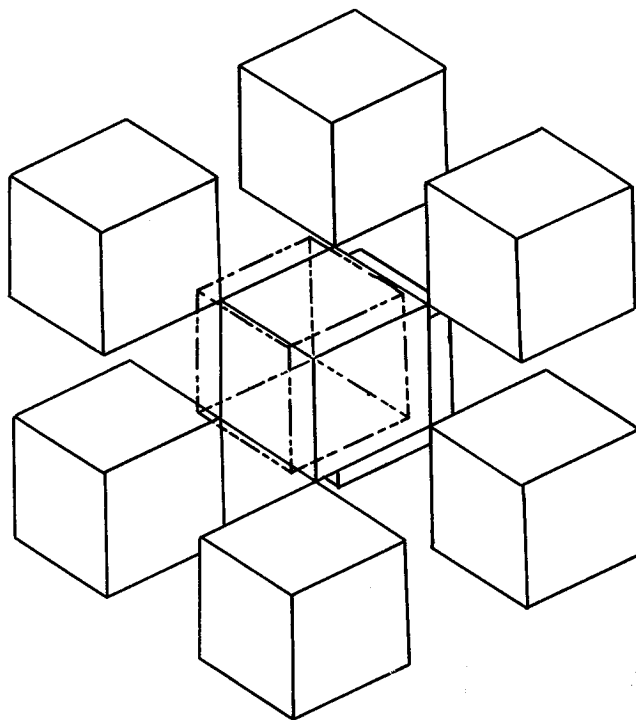
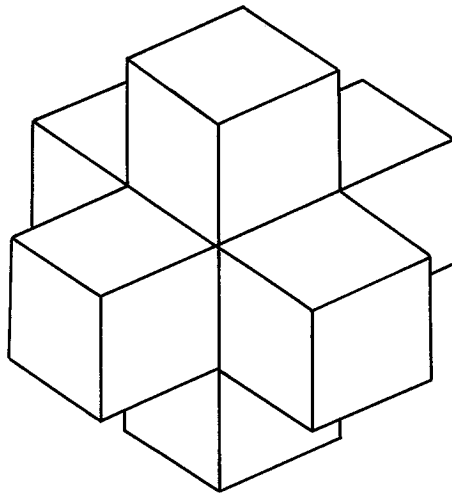


Figure 2.37

A similar transformation with close packed polyhedron may also be done when each cell transforms by the geometric transformation concepts discussed earlier. The close packed units transform however, faces are shared with adjacent polyhedra thus creating new unit cells in a close packed state. Figure 2.38 illustrates the face transformation concept of this type.

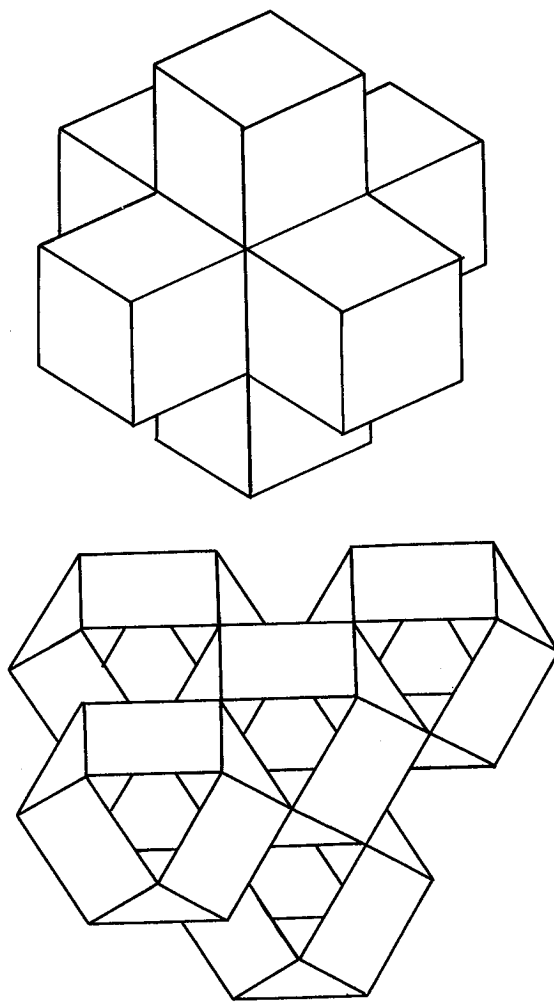


Figure 2.38

Part III

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